

SCHRÖDINGER OPERATOR WITH PERIODIC PLUS COMPACTLY SUPPORTED POTENTIALS ON THE HALF-LINE

EVGENY KOROTYAEV

ABSTRACT. We consider the Schrödinger operator H with a periodic potential p plus a compactly supported potential q on the half-line. We prove the following results: 1) a forbidden domain for the resonances is specified, 2) asymptotics of the resonance-counting function is determined, 3) in each nondegenerate gap γ_n for n large enough there is exactly an eigenvalue or an antibound state, 4) the asymptotics of eigenvalues and antibound states are determined at high energy, 5) the number of eigenvalues plus antibound states is odd ≥ 1 in each gap, 6) between any two eigenvalues there is an odd number ≥ 1 of antibound states, 7) for any potential q and for any sequences $(\sigma_n)_1^\infty, \sigma_n \in \{0, 1\}$ and $(\varkappa_n)_1^\infty \in \ell^2, \varkappa_n \geq 0$, there exists a potential p such that each gap length $|\gamma_n| = \varkappa_n, n \geq 1$ and H has exactly σ_n eigenvalues and $1 - \sigma_n$ antibound state in each gap $\gamma_n \neq \emptyset$ for n large enough, 8) if unperturbed operator (at $q = 0$) has infinitely many virtual states, then for any sequence $(\sigma)_1^\infty, \sigma_n \in \{0, 1\}$, there exists a potential q such that H has σ_n bound states and $1 - \sigma_n$ antibound states in each gap open γ_n for n large enough.

1. INTRODUCTION AND MAIN RESULTS

Consider the Schrödinger operator H acting in the Hilbert space $L^2(\mathbb{R}_+)$ and given by

$$H = H_0 + q, \quad H_0 f = -f'' + p(x)f, \quad f(0) = 0, \\ p \in L^1_{real}(\mathbb{R}/\mathbb{Z}), \quad q \in \mathcal{Q}_t = \{q : q \in L^2_{real}(\mathbb{R}_+), \sup(\text{supp}(q)) = t\}, \quad t > 0. \quad (1.1)$$

The spectrum of H_0 consists of an absolutely continuous part $\sigma_{ac}(H_0) = \cup_{n \geq 1} \mathfrak{S}_n$ plus at most one eigenvalue in each gap $\gamma_n \neq \emptyset, n \geq 1$ ([CL], [E], [Zh3]), where the bands \mathfrak{S}_n and gaps γ_n are given by (see Fig. 1)

$$\mathfrak{S}_n = [E_{n-1}^+, E_n^-], \quad \gamma_n = (E_n^-, E_n^+), \quad n \geq 1, \quad E_0^+ < \dots \leq E_{n-1}^+ < E_n^- \leq E_n^+ < \dots$$

Let below $E_0^+ = 0$. The sequence $E_0^+ < E_1^- \leq E_1^+ < \dots$ is the spectrum of the equation

$$-y'' + p(x)y = \lambda y, \quad \lambda \in \mathbb{C}, \quad (1.2)$$

with the 2-periodic boundary conditions, i.e. $y(x+2) = y(x), x \in \mathbb{R}$. The bands $\mathfrak{S}_n, \mathfrak{S}_{n+1}$ are separated by a gap γ_n and let $\gamma_0 = (-\infty, E_0^+)$. If a gap degenerates, that is $\gamma_n = \emptyset$, then the corresponding bands \mathfrak{S}_n and \mathfrak{S}_{n+1} merge. If $E_n^- = E_n^+$ for some n , then this number E_n^\pm is the double eigenvalue of equation (1.2) with the 2-periodic boundary conditions. The lowest eigenvalue $E_0^+ = 0$ is always simple and the corresponding eigenfunction is 1-periodic. The eigenfunctions, corresponding to the eigenvalue E_{2n}^\pm , are 1-periodic, and for the case E_{2n+1}^\pm they are anti-periodic, i.e., $y(x+1) = -y(x), x \in \mathbb{R}$.

Date: May 7, 2009.

1991 *Mathematics Subject Classification.* 34A55, (34B24, 47E05).

Key words and phrases. resonances, scattering, periodic potential, S-matrix.

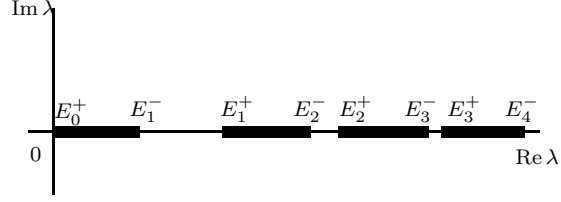


FIGURE 1. The cut domain $\mathbb{C} \setminus \cup \mathfrak{S}_n$ and the cuts (bands) $\mathfrak{S}_n = [E_{n-1}^+, E_n^-]$, $n \geq 1$

We describe properties the operator $\mathcal{H}y = -f'' + (p + q)y$ on the real line, where p is periodic and q is compactly supported. The spectrum of \mathcal{H} consists of an absolutely continuous part $\sigma_{ac}(\mathcal{H}) = \sigma_{ac}(H_0)$ plus a finite number of simple eigenvalues in each gap $\gamma_n \neq \emptyset$, $n \geq 0$, see [Rb], [F1] and at most two eigenvalue [Rb] in every open gap γ_n for n large enough. If $q_0 = \int_{\mathbb{R}} q(x)dx \neq 0$, then \mathcal{H} has precisely one eigenvalue (see [Zh1], [F2], [GS]) and one antibound state [K4] in each gap $\gamma_n \neq \emptyset$ for n large enough. If $q_0 = 0$, then roughly speaking there are two eigenvalues and zero antibound state or zero eigenvalues and two antibound states [K4] in each gap $\gamma_n \neq \emptyset$ for n large enough.

The spectrum of H acting in $L^2(\mathbb{R}_+)$ consists of an absolutely continuous part $\sigma_{ac}(H) = \sigma_{ac}(H_0)$ plus a finite number of simple eigenvalues in each gap $\gamma_n \neq \emptyset$, $n \geq 0$. Note that the last fact follows from the same result for \mathcal{H} and the splitting principle.

Let $\varphi(x, z), \vartheta(x, z)$ be the solutions of the equation $-y'' + py = z^2y$ satisfying $\varphi'(0, z) = \vartheta(0, z) = 1$ and $\varphi(0, z) = \vartheta'(0, z) = 0$, where $y' = \partial_x y$. The Lyapunov function is defined by $\Delta(z) = \frac{1}{2}(\varphi'(1, z) + \vartheta(1, z))$. The function $\Delta^2(\sqrt{\lambda})$ is entire, where $\sqrt{\lambda}$ is defined by $\sqrt{1} = 1$. Introduce the function $(1 - \Delta^2(\sqrt{\lambda}))^{\frac{1}{2}}$, $\lambda \in \overline{\mathbb{C}}_+$ and we fix the branch by the condition $(1 - \Delta^2(\sqrt{\lambda + i0}))^{\frac{1}{2}} > 0$ for $\lambda \in \mathfrak{S}_1 = [E_0^+, E_1^-]$. Introduce the two-sheeted Riemann surface Λ of $(1 - \Delta^2(\sqrt{\lambda}))^{\frac{1}{2}}$ obtained by joining the upper and lower rims of two copies of the cut plane $\mathbb{C} \setminus \sigma_{ac}(H_0)$ in the usual (crosswise) way. The n -th gap on the first physical sheet Λ_1 we will denote by $\gamma_n^{(1)}$ and the same gap but on the second nonphysical sheet Λ_2 we will denote by $\gamma_n^{(2)}$ and let γ_n^c be the union of $\overline{\gamma}_n^{(1)}$ and $\overline{\gamma}_n^{(2)}$:

$$\gamma_n^c = \overline{\gamma}_n^{(1)} \cup \overline{\gamma}_n^{(2)}. \quad (1.3)$$

It is well known that the function $f(\lambda) = ((H - \lambda)^{-1}h, h)$ has meromorphic extension from the physical sheet Λ_1 into the Riemann surface Λ for each $h \in C_0^\infty(\mathbb{R}_+)$, $h \neq 0$. Moreover, if f has a pole at some $\lambda_0 \in \Lambda_1$ and some h , then λ_0 is an eigenvalue of H and $\lambda_0 \in \cup \gamma_n^{(1)}$.

Definition Λ . Let $f(\lambda) = ((H - \lambda)^{-1}h, h)$, $\lambda \in \Lambda$ for some $h \in C_0^\infty(\mathbb{R}_+)$, $h \neq 0$.

- 1) If $f(\lambda)$ has a pole at some $\lambda_0 \in \Lambda_2$, $\lambda_0 \neq E_n^\pm$, $n \geq 0$, we say that λ_0 is a resonance.
- 2) A point $\lambda_0 = E_n^\pm$, $n \geq 0$ is a virtual state, if the function $f(\lambda_0 + z^2)$ has a pole at 0.
- 3) A point $\lambda \in \Lambda$ is a state, if it is a bound state or a resonance or a virtual state. Let $\mathfrak{S}_{st}(H)$ be the set of all states. The multiplicity of the state is the multiplicity of the corresponding pole. If $\lambda_0 \in \gamma_n^{(2)}$, $n \geq 0$, then we call λ_0 an antibound state.

As a good example we consider the states of the operator H_0 for the case $p \neq \text{const}$, $q = 0$, see [Zh3], [HKS]. Let $f_0(\lambda) = ((H_0 - \lambda)^{-1}h, h)$ for some $h \in C_0^\infty(\mathbb{R}_+)$. It is well known that the function f_0 is meromorphic on the physical sheet Λ_1 and has a meromorphic extension into Λ . For each $\gamma_n^c \neq \emptyset$, $n \geq 1$ there is exactly one state $\lambda_n^0 \in \gamma_n^c$ of H_0 and its projection

on the complex plane coincides with the Dirichlet eigenvalues μ_n^2 . Moreover, there is one case from three ones:

- 1) $\lambda_n^0 \in \gamma_n^{(1)}$ is an eigenvalues,
- 2) $\lambda_n^0 \in \gamma_n^{(2)}$ is an antibound state,
- 3) $\lambda_n^0 = E_n^+$ or $\lambda_n^0 = E_n^-$ is a virtual states.

There are no other states of H_0 . Thus H_0 has only eigenvalues, virtual states and antibound states. If there are exactly $N \geq 1$ nondegenerate gaps in the spectrum of $\sigma_{ac}(H_0)$, then operator H_0 has exactly N states. The gaps $\gamma_n = \emptyset$ do not give contribution to the states. In particular, if all $\gamma_n = \emptyset, n \geq 1$, then $p = 0$, see [MO] or [K4] and H_0 has not states. The states λ_n^0 are described in Lemma 2.1.

We need the results about the inverse spectral theory for the unperturbed operator H_0 : define the mapping $p \rightarrow \xi = (\xi_n)_1^\infty$, where the components $\xi_n = (\xi_{1n}, \xi_{2n}) \in \mathbb{R}^2$ are given by

$$\xi_{1n} = \frac{E_n^- + E_n^+}{2} - \mu_n^2, \quad \xi_{2n} = \left| \frac{|\gamma_n|^2}{4} - \xi_{1n}^2 \right|^{\frac{1}{2}} a_n, \quad a_n = \begin{cases} +1 & \text{if } \lambda_n^0 \text{ is an eigenvalues} \\ -1 & \text{if } \lambda_n^0 \text{ is a resonance} \\ 0 & \text{if } \lambda_n^0 \text{ is a virtual state} \end{cases}.$$

Recall the results from [K5]: *The mapping $\xi : \mathcal{H} \rightarrow \ell^2 \oplus \ell^2$ is a real analytic isomorphism between real Hilbert spaces $\mathcal{H} = \{p \in L^2(0, 1) : \int_0^1 p(x)dx = 0\}$ and $\ell^2 \oplus \ell^2$ and the estimates hold true*

$$\|p\| \leq 4\|\xi\|(1 + \|\xi\|^{\frac{1}{3}}), \quad \|\xi\| \leq \|p\|(1 + \|p\|)^{\frac{1}{3}}, \quad (1.4)$$

where $\|p\|^2 = \int_0^1 p^2(x)dx$ and $\|\xi\|^2 = \frac{1}{4} \sum |\gamma_n|^2$. Estimates (1.4) were proved in [K7]. Moreover, for any sequence $\varkappa = (\varkappa_n)_1^\infty \in \ell^2, \varkappa_n \geq 0$ there are unique 2-periodic eigenvalues $E_n^\pm, n \geq 0$ for some $p \in \mathcal{H}$ such that each $\varkappa_n = E_n^+ - E_n^-, n \geq 1$. Thus if we know gap lengths $(|\gamma_n|)_1^\infty$, then we can recover the Riemann surface Λ uniquely plus the points $E_n^- = E_n^+$, if $\varkappa_n = 0$. Furthermore, for any sequence $\tilde{\lambda}_n^0 \in \gamma_n^c, n \geq 1$, there is an unique potential $p \in \mathcal{H}$, such that each state λ_n^0 (corresponding to p) coincides with $\tilde{\lambda}_n^0, n \geq 1$. Remark that results of [K5] were generalized in [K6] for periodic distributions $p = w'$, where $w \in \mathcal{H}$.

Define the function

$$D(\lambda) = \det(I + q(H_0 - \lambda)^{-1}), \quad \lambda \in \mathbb{C}_+.$$

It is well known that the function $D(\lambda)$ is analytic in $\lambda \in \mathbb{C}_+$ and has a meromorphic extension into Λ . Each zero of $D(\lambda)$ in Λ_1 is an eigenvalue of H and belongs to the union of physical gaps $\cup_{n \geq 0} \gamma_n^{(1)}$. Until now only some particular results are known about the zeros on the nonphysical sheet Λ_2 . Remark that the set of zeros of D on Λ_2 is symmetric with respect to the real line, since D is real on $\gamma_0^{(2)}$.

Let $\Phi(x, z)$ be the fundamental solution of the equation

$$-\Phi'' + (p + q)\Phi = z^2\Phi, x \geq 0, \quad \Phi(0, z) = 0, \quad \Phi'(0, z) = 1, \quad z \in \mathbb{C}. \quad (1.5)$$

Theorem 1.1. *i) Let $\varepsilon > 0$. The function D satisfies*

$$D(\lambda) = 1 + \frac{\widehat{q}(z) - \widehat{q}(0)}{2iz} + \frac{O(e^{t(|\operatorname{Im} z| - \operatorname{Im} z)})}{z^2} \quad \text{as } |z| \rightarrow \infty, z = \sqrt{\lambda}, \sqrt{1} = 1, \quad (1.6)$$

where $|\lambda - E_n^\pm| \geq n\varepsilon$ for all $n \geq 1$ and $\widehat{q}(z) = \int_0^t q(x)e^{2izx}dx$ and

$$\mathfrak{S}_{st}(H) \setminus \mathfrak{S}_{st}(H_0) = \{\lambda \in \Lambda \setminus \mathfrak{S}_{st}(H_0) : D(\lambda) = 0\} \subset \Lambda_2 \cup \bigcup_{n \geq 0} \gamma_n^{(1)}, \quad (1.7)$$

$$\mathfrak{S}_{st}(H) \cap \mathfrak{S}_{st}(H_0) = \mathfrak{S}_{st}(H_0) \cap \{z \in \sigma_{st}(H_0) : \Phi(n_t, z) = 0\}, \quad n_t = \inf_{n \in \mathbb{N}, n \geq t} n. \quad (1.8)$$

ii) If $\lambda_n^0 \in \mathfrak{S}_{st}(H) \cap \mathfrak{S}_{st}(H_0)$, then $\lambda_n^0 \in \bar{\gamma}_n^{(j)} \neq \emptyset$ for some $j = 1, 2, n \geq 1$ and

$$D(\lambda) \rightarrow D(\lambda_n^0) \neq 0 \quad \text{as} \quad \lambda \rightarrow \lambda_n^0. \quad (1.9)$$

iii) (The logarithmic law.) Each resonance $\lambda \in \Lambda_2$ of H satisfies

$$|\sqrt{\lambda} \sin \sqrt{\lambda}| \leq C_F e^{(2t+1)|\operatorname{Im} \sqrt{\lambda}|}, \quad C_F = 3(\|p\|_1 + \|p+q\|_t) e^{2\|p+q\|_t + \|p\|_1}, \quad (1.10)$$

and there are no resonances in the domain $\mathcal{D}_{forb} = \{\lambda \in \Lambda_2 \setminus \cup \bar{\gamma}_n^{(2)} : 4C_F e^{2|\operatorname{Im} \sqrt{\lambda}|} < |\lambda|^{\frac{1}{2}}\}$.

Remark. 1) Let $\lambda_n^0 \in \gamma_n^{(1)}$ be a eigenvalue of H_0 for some $n \geq 1$. If $\Phi(n_t, \mu_n) = 0$, then μ_n^2 is a Dirichlet eigenvalue of the problem $-y'' + (p+q)y = \mu_n^2 y, y(0) = y(n_t) = 0$. Then by (1.8), λ_n^0 is a bound state of H and (1.9) yields $D(\lambda_n^0) \neq 0$. Thus λ_n^0 is a **pole of a resolvent**, but λ_n^0 is **neither a zero of D nor a pole** of the S-matrix for H, H_0 given by

$$\mathcal{S}_M(z) = \frac{\overline{D(\lambda)}}{D(\lambda)}, \quad \lambda \in \sigma_{ac}(H_0). \quad (1.11)$$

2) If $D(\lambda) = 0$ for some $\lambda = E_n^\pm \neq \mu_n^2, n \geq 0$, then by (1.7), λ is a virtual state.

3) If $\mu_n^2 = E_n^\pm$ for some $n \geq 1$, then by (1.8), μ_n^2 is a virtual state iff $\Phi(n_t, \mu_n) = 0$.

4) If $p = 0$, then it is well known that each zero of $D(\cdot)$ is a state, see e.g., [K1], [S]. Moreover, each resonance lies below a logarithmic curve (depending only in q see e.g. [K1], [Z]). The forbidden domain $\mathcal{D}_{forb} \cap \mathbb{C}_-$ is similar to the case $p = 0$, see [K1].

Let $\#(H, r, A)$ be the total number of state of H in the set $A \subseteq \Lambda$ having modulus $\leq r$, each state being counted according to its multiplicity.

Define the Fourier coefficients p_{sn}, \hat{q}_{cn} and the Fourier transform \hat{q} by

$$q_0 = \int_0^1 q(x) dx, \quad p_{sn} = \int_0^1 p(x) \sin 2\pi n x dx, \quad \hat{q}(z) = \int_0^t q(x) e^{2izx} dx, \quad \hat{q}_{cn} = \operatorname{Re} \hat{q}(\pi n). \quad (1.12)$$

Theorem 1.2. i) H has an odd number ≥ 1 of states on each set $\gamma_n^c \neq \emptyset, n \geq 1$, where γ_n^c is a union of the physical $\bar{\gamma}_n^{(1)} \subset \Lambda_1$ and non-physical gap $\bar{\gamma}_n^{(2)} \subset \Lambda_2$ and H has exactly one simple state $\lambda_n \in \gamma_n^c$ for all $n \geq 1 + 4C_F e^{t^{\frac{5}{2}}}$ with asymptotics

$$\sqrt{\lambda_n} = \mu_n - \frac{(q_0 - \hat{q}_{cn})p_{sn} + O(\frac{1}{n})}{2(\pi n)^2} \quad \text{as} \quad n \rightarrow \infty. \quad (1.13)$$

Moreover, the following asymptotics hold true as $r \rightarrow \infty$:

$$\#(H, r, \Lambda_2 \setminus \cup \gamma_n^{(2)}) = r \frac{2t + o(1)}{\pi}, \quad (1.14)$$

$$\#(H, r, \mathbb{R}) = \#(H_0, r, \mathbb{R}) + 2N_q \quad \text{for some integer } N_q \geq 0, \quad r \notin \cup \bar{\gamma}_n. \quad (1.15)$$

ii) Let λ be an eigenvalue of H and let $\lambda^{(2)} \in \Lambda_2$ be the same number but on the second sheet Λ_2 . Then $\lambda^{(2)}$ is not an anti-bound state.

iii) Let $\lambda_1 < \lambda_2$ and let $\lambda_1, \lambda_2 \in \gamma_n^{(1)}$ be some eigenvalues of H for some $n \geq 0$ and assume that there are no other eigenvalues on the interval $\Omega = (\lambda_1, \lambda_2) \subset \gamma_n^{(1)}$. Let $\Omega^{(2)} \subset \gamma_n^{(2)} \subset \Lambda_2$ be the same interval but on the second sheet. Then there exists an odd number ≥ 1 of antibound states on $\Omega^{(2)}$.

Remark. 1) Results (iii) at $p = 0$ were obtained independently in [K1], [S].

2) First term in the asymptotics (1.14) does not depend on the periodic potential p . Recall that asymptotics (1.14) for the case $p = 0$ was obtained by Zworski [Z].

2) The main difference between the distribution of the resonances for the case $p \neq \text{const}$ and $p = \text{const}$ is the bound states and antibound states in high energy gaps, see (1.13).

3) In the proof of (1.14) we use the Paley Wiener type Theorem from [Fr], the Levinson Theorem (see Sect. 4) and analysis of the function D near λ_n^0 .

4) For even potentials $p \in L_{\text{even}}^2(0, 1) = \{p \in L^2(0, 1), p(x) = p(1-x), x \in (0, 1)\}$ all coefficients $p_{sn} = 0$ and asymptotics (1.13) are not sharp. This case is described in Theorem 1.4.

5) Assume that a potential $u \in L^2(\mathbb{R}_+)$ is compactly supported, $\text{supp } u \subset (0, t)$ and satisfies $|\hat{u}_n| = o(n^{-\alpha})$ as $n \rightarrow \infty$. Then in the case (ii) the operator $H + u$ has the same number of bound states in each gap $\gamma_n \neq \emptyset$ for n large enough.

We consider a stability of real states λ_n . Recall that $\lambda_n^0 \in \gamma_n^c$ is a state of H_0 .

Theorem 1.3. *Let $b_n = q_0 - \hat{q}_{cn}$. Assume that $|p_{sn}| > n^{-\alpha}$ and $|b_n| > n^{-(1-\alpha)}$ for some $\alpha \in (0, 1)$ and for all $n \in \mathbb{N}_0$, where $\mathbb{N}_0 \subset \mathbb{N}$ is some infinite subset such that each $|\gamma_n| > 0, n \in \mathbb{N}_0$. Let $b_n > 0$ (or $b_n < 0$). Then the real state $\lambda_n \in \gamma_n^c$ for $n \in \mathbb{N}_0$ large enough satisfies:
If λ_n^0 is an eigenvalue of H_0 , then λ_n is an eigenvalue of H and $\lambda_n^0 < \lambda_n$ (or $\lambda_n^0 > \lambda_n$).
If λ_n^0 is an antibound state of H_0 , then λ_n is an antibound state of H and $\lambda_n < \lambda_n^0$ (or $\lambda_n^0 < \lambda_n$).*

Remark. 1) Let $q > 0$. It is well known that if the coupling constant $\tau > 0$ is increasing, then eigenvalues of $H_0 + \tau q$ are increasing too. Roughly speaking, in our case the antibound states in the gap move in opposite direction.

2) We explain roughly transformations: resonances \rightarrow antibound states \rightarrow bound states. Consider the operator $H_\tau = H_0 + \tau q$, where $\tau \in \mathbb{R}$ is the coupling constant. If $\tau = 0$, then H_0 has only states $\lambda_n^0, n \geq 1$ (eigenvalues, antibound states and virtual states). Consider the first gap $\gamma_1^c \neq \emptyset$. If τ is increasing, then the state λ_1^0 moves and there are no other states on γ_1^c . If τ is increasing again, then λ_1^0 removes on the physical gap $\gamma_1^{(1)}$ and becomes an eigenvalue; there are no new eigenvalues but some two complex resonances ($\lambda \in \mathbb{C}_+ \subset \Lambda_2$ and $\bar{\lambda} \in \mathbb{C}_- \subset \Lambda_2$) reach non physical gap $\gamma_1^{(2)}$ and transform into new antibound states. If τ is increasing again, then some new antibound states will be virtual states, and then later they will be bound states. Thus if τ runs through \mathbb{R}_+ , then there is a following transformation: resonances \rightarrow antibound states \rightarrow virtual states \rightarrow bound states \rightarrow virtual states...

Recall that $p \in L_{\text{even}}^2(0, 1)$ iff $\mu_n^2 = E_n^-$ or $\mu_n^2 = E_n^+$ for all $n \geq 1$, see [GT], [KK1].

Theorem 1.4. *i) Let unperturbed states $\lambda_n^0 \in \{E_n^-, E_n^+\}$ for all $n \in \mathbb{N}_0$ large enough, where $\mathbb{N}_0 \subset \mathbb{N}$ is some infinite subset such that each $|\gamma_n| > 0, n \in \mathbb{N}_0$. Then*

$$\sqrt{\lambda_n} = \mu_n + s_n |\gamma_n| \frac{(q_0 - \hat{q}_{cn} + O(\frac{1}{n}))^2}{(2\pi n)^2}, \quad s_n = \begin{cases} + & \text{if } \mu_n^2 = E_n^- \\ - & \text{if } \mu_n^2 = E_n^+ \end{cases}, \quad n \in \mathbb{N}_0 \quad (1.16)$$

as $n \rightarrow \infty$. Moreover, if $\alpha \in (\frac{1}{2}, 1)$, then for $n \in \mathbb{N}_0$ large enough the following holds true:

if $\lambda_n^0 = E_n^-$, $q_0 - \hat{q}_{cn} > n^{-\alpha}$ or $\lambda_n^0 = E_n^+$, $q_0 - \hat{q}_{cn} < -n^{-\alpha}$, then λ_n is an eigenvalue,
if $\lambda_n^0 = E_n^-$, $q_0 - \hat{q}_{cn} < -n^{-\alpha}$ or $\lambda_n^0 = E_n^+$, $q_0 - \hat{q}_{cn} > n^{-\alpha}$, then λ_n is an antibound state.
ii) Let $q \in \mathcal{Q}_t$ satisfy $|q_0 - \hat{q}_{cn}| > n^{-\alpha}$ for n large enough and for some $\alpha \in (\frac{1}{2}, 1)$. Then for any sequences $(\sigma_n)_1^\infty, \sigma_n \in \{0, 1\}$ and $(\varkappa_n)_1^\infty \in \ell^2, \varkappa_n \geq 0$ there exists a potential $p \in L^2(0, 1)$ such

that each gap length $|\gamma_n| = \varkappa_n, n \geq 1$ and H has exactly σ_n eigenvalues and $1 - \sigma_n$ antibound states in each gap $\gamma_n \neq \emptyset$ for n large enough.

iii) Let $p \in L^1(0, 1)$ and let unperturbed states $\lambda_n^0 \in \{E_n^-, E_n^+\}$ for all $n \in \mathbb{N}_0$ large enough, where $\mathbb{N}_0 \subset \mathbb{N}$ is some infinite subset such that each $|\gamma_n| > 0, n \in \mathbb{N}_0$. Then for any sequence $(\sigma_n)_1^\infty, \sigma_n \in \{0, 1\}$ there exists a potential $q \in \mathcal{Q}_t$ such that H has exactly σ_n eigenvalues and $1 - \sigma_n$ antibound states in each gap $\gamma_n \neq \emptyset$ for $n \in \mathbb{N}_0$ large enough.

Remark. Roughly speaking (1.16) is asymptotics for even potentials $p \in L^1(0, 1)$.

Let $\#_{bs}(H, \Omega)$ (or $\#_{abs}(H, \Omega)$) be the total number of bound states (or anti bound states) of H on the segment $\Omega \subset \overline{\gamma}_n^{(1)} \subset \Lambda_1$ (or $\Omega \subset \overline{\gamma}_n^{(2)} \subset \Lambda_2$) for some $n \geq 0$ (each antibound state being counted according to its multiplicity).

Introduce the integrated density of states $\rho(\lambda), \lambda \in \mathbb{R}$ (a continuous function on \mathbb{R}) by

$$\rho(\lambda)|_{\gamma_n} = n, \quad \rho(\mathfrak{S}_{n+1}) = [n, n+1] \quad \cos \pi \rho(\lambda) = \Delta(\sqrt{\lambda}), \quad \lambda \in \mathfrak{S}_{n+1}, \quad n \geq 0. \quad (1.17)$$

The real function ρ is strongly increasing on each spectral band \mathfrak{S}_n . It is well known that $\rho(\lambda) = \frac{1}{\pi} \operatorname{Re} k(\sqrt{\lambda + i0}), \lambda \in \mathbb{R}$, where k is the quasimomentum defined in Section 2.

Corollary 1.5. Let $H_\tau = H_0 + q_\tau$ where $q_\tau = q(\frac{x}{\tau}), \tau \geq 1$. Let $\Omega = [E_1, E_2] \subset \overline{\gamma}_n^{(1)} \neq \emptyset$ be some interval on the physical sheet Λ_1 for some $n \geq 0$ and let $\Omega^{(2)} \subset \overline{\gamma}_n^{(2)}$ be the same interval, but on the non-physical sheet Λ_2 . Then

$$\begin{aligned} \#_{abs}(H_\tau, \Omega^{(2)}) &\geq 1 + \#_{bs}(H_\tau, \Omega) = \\ &\tau \int_0^\infty \left(\rho(E_2 - q(x)) - \rho(E_1 - q(x)) \right) dx + o(\tau) \quad \text{as} \quad \tau \rightarrow \infty. \end{aligned} \quad (1.18)$$

Remark. 1) In the proof of (1.18) we use the Sobolev's results [So] about asymptotics of $\#_{bs}(H_\tau, \Omega)$ with needed modifications of Schmidt [Sc]. Sobolev considered the case $H_\tau = H_0 + \tau V$, where $V(x) = \frac{c+o(1)}{x^\alpha}$ as $x \rightarrow \infty$, for some $c \neq 0, \alpha > 0$. We can not apply this results to our case, since this potential V is not compactly supported. We use the modification of Schmidt for the perturbation of the periodic Dirac operator, where the decreasing potential can be compactly supported.

A lot of papers are devoted to the resonances for the Schrödinger operator with $p = 0$, see [Fr], [H], [K1], [K2], [S], [Z] and references therein. Although resonances have been studied in many settings, but there are relatively few cases where the asymptotics of the resonance counting function are known, mainly one dimensional case [Fr], [K1], [K2], [S], and [Z]. We recall that Zworski [Z] obtained the first results about the distribution of resonances for the Schrödinger operator with compactly supported potentials on the real line. The author obtained the characterization (plus uniqueness and recovering) of S -matrix for the Schrödinger operator with a compactly supported potential on the real line [K2] and the half-line [K1], see also [Z1], [BKW] about uniqueness.

The author [K3] obtained the stability results for the Schrödinger operator on the half line:

(i) If $\varkappa = (\varkappa)_1^\infty$ is a sequence of poles (eigenvalues and resonances) of the S -matrix for some real compactly supported potential q and $\tilde{\varkappa} - \varkappa \in \ell_\varepsilon^2$ for some $\varepsilon > 1$, then $\tilde{\varkappa}$ is the sequence of poles of the S -matrix for some unique real compactly supported potential \tilde{q} .

(ii) The measure associated with the poles of the S -matrix is the Carleson measure, the sum $\sum (1 + |\varkappa_n|)^{-\alpha}, \alpha > 1$ is estimated in terms of the L^1 -norm of the potential q .

Brown and Weikard [BW] considered the Schrödinger operator $-y'' + (p_A + q)y$ on the half-line, where p_A is an algebro-geometric potentials and q is a compactly supported potential. They proved that the zeros of the Jost function determine q uniquely.

Christiansen [Ch] considered resonances associated to the Schrödinger operator $-y'' + (p_S + q)y$ on the real line, where p_S is a step potential. She determined asymptotics of the resonance-counting function. Moreover, she obtained that the resonances determine q uniquely.

Describe recent author's results [K4] about the operator $\mathcal{H} = \mathcal{H}_0 + q$, $\mathcal{H}_0 = -\frac{d^2}{dx^2} + p$ on the real line, where p is periodic and q is compactly supported: 1) asymptotics of the resonance-counting function is determined, 2) a forbidden domain for the resonances is specified, 3) the asymptotics of eigenvalues and antibound states are determined, 4) for any sequence $(\sigma)_1^\infty, \sigma_n \in \{0, 2\}$, there exists a compactly supported potential q such that \mathcal{H} has σ_n bound states and $2 - \sigma_n$ antibound states in each gap $\gamma_n \neq \emptyset$ for n large enough, 5) for any q (with $q_0 = 0$) and for any sequences $(\sigma_n)_1^\infty, \sigma_n \in \{0, 2\}$ and $(\varkappa_n)_1^\infty \in \ell^2, \varkappa_n \geq 0$ there exists a potential $p \in L^2(0, 1)$ such that each gap length $|\gamma_n| = \varkappa_n, n \geq 1$ and \mathcal{H} has exactly σ_n eigenvalues and $2 - \sigma_n$ antibound states in each gap $\gamma_n \neq \emptyset$ for n large enough.

We compare the results for \mathcal{H} on \mathbb{R} and H on \mathbb{R}_+ : 1) their properties are close for even potentials $p \in L_{even}^2(0, 1)$, since in this case unperturbed operators H_0 have only virtual states, 2) if p is not even, then the unperturbed operator H_0 (in general) has eigenvalues, virtual states and antibound states, but the operator \mathcal{H}_0 has exactly two virtual states in each open gap. This leads to the different properties of H, \mathcal{H} and roughly speaking the case of H is more complicated, since the unperturbed operator H_0 is more complicated.

The plan of the paper is as follows. In Section 2 we define the Riemann surface associated the momentum variable $z = \sqrt{\lambda}, \lambda \in \Lambda$, and describe the preliminary results about fundamental solutions. In Sections 3 we study states of H . In Sections 4 we prove the main Theorem 1.1-1.4. In the proof of theorems we use properties of the quasimomentum, a priori estimates from [KK], [MO], and results from the inverse theory for the Hill operator from [K5]. In the proof the analysis of the function $F(z) = \varphi(1, z)D(z)\overline{D}(z), z^2 \in \sigma_{ac}(H)$ is important, since we obtain the relationship between zeros of F (which is entire) and the states. Thus we reduce the spectral problems of H to the problem of entire function theory.

2. PRELIMINARIES

We will work with the momentum $z = \sqrt{\lambda}$, where λ is an energy and recall that $\sqrt{1} = 1$. Introduce the cut domain (see Fig.2)

$$\mathcal{Z} = \mathbb{C} \setminus \cup \overline{g}_n, \quad \text{where } g_n = (e_n^-, e_n^+) = -g_{-n}, \quad e_n^\pm = \sqrt{E_n^\pm} > 0, \quad n \geq 1. \quad (2.1)$$

Note that $\Delta(e_n^\pm) = (-1)^n$. If $\lambda \in \gamma_n, n \geq 1$, then $z \in g_{\pm n}$ and if $\lambda \in \gamma_0 = (-\infty, 0)$, then $z \in g_0^\pm = i\mathbb{R}_\pm$. Below we will use the momentum variable $z = \sqrt{\lambda}$ and the corresponding Riemann surface \mathcal{M} , which is more convenient for us, than the Riemann surface Λ . Slitting the n -th momentum gap g_n (suppose it is nontrivial) we obtain a cut g_n^c with an upper g_n^+ and lower rim g_n^- . Below we will identify this cut g_n^c and the union of of the upper rim (gap) \overline{g}_n^+ and the lower rim (gap) \overline{g}_n^- , i.e.,

$$g_n^c = \overline{g}_n^+ \cup \overline{g}_n^-, \quad \text{where } g_n^\pm = g_n \pm i0; \quad \text{and if } z \in g_n \Rightarrow z \pm i0 \in g_n^\pm. \quad (2.2)$$

In order to construct the Riemann surface \mathcal{M} we take the cut domain $\mathcal{Z} = \mathbb{C} \setminus \cup \overline{g}_n$ and identify (we glue) the upper rim g_n^+ of the slit g_n^c with the upper rim g_{-n}^+ of the slit g_n^c and correspondingly the lower rim g_n^- of the slit g_n^c with the lower rim g_{-n}^- of the slit g_n^c for

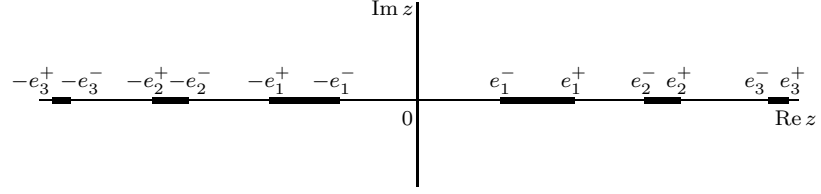


FIGURE 2. The cut domain $\mathcal{Z} = \mathbb{C} \setminus \bigcup \overline{g_n}$ and the slits $g_n = (e_n^-, e_n^+)$ in the z -plane.

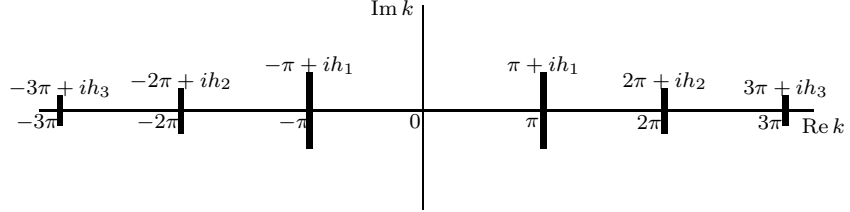


FIGURE 3. The domain $\mathcal{K} = \mathbb{C} \setminus \bigcup \Gamma_n$, where the slit $\Gamma_n = (\pi n - ih_n, \pi n + ih_n)$

all nontrivial gaps. The mapping $z = \sqrt{\lambda}$ from Λ onto \mathcal{M} is one-to-one and onto. The gap $\gamma_n^{(1)} \subset \Lambda_1$ is mapped onto $g_n^+ \subset \mathcal{M}_1$ and the gap $\gamma_n^{(2)} \subset \Lambda_2$ is mapped onto $g_n^- \subset \mathcal{M}_2$. From a physical point of view, the upper rim g_n^+ is a physical gap and the lower rim g_n^- is a non physical gap. Moreover, $\mathcal{M} \cap \mathbb{C}_+ = \mathcal{Z} \cap \mathbb{C}_+$ plus all physical gaps g_n^+ is a so-called physical "sheet" \mathcal{M}_1 and $\mathcal{M} \cap \mathbb{C}_- = \mathcal{Z} \cap \mathbb{C}_-$ plus all non physical gaps g_n^- is a so-called non physical "sheet" \mathcal{M}_2 . The set (the spectrum) $\mathbb{R} \setminus \bigcup g_n$ joints the first \mathcal{M}_1 and second sheets \mathcal{M}_2 .

We introduce the quasimomentum $k(\cdot)$ for H_0 by $k(z) = \arccos \Delta(z)$, $z \in \mathcal{Z}$. The function $k(z)$ is analytic in $z \in \mathcal{Z}$ and satisfies:

$$(i) \quad k(z) = z + O(1/z) \quad \text{as } |z| \rightarrow \infty, \quad (ii) \quad \operatorname{Re} k(z \pm i0)|_{[e_n^-, e_n^+]} = \pi n, \quad n \in \mathbb{Z}, \quad (2.3)$$

and $\pm \operatorname{Im} k(z) > 0$ for any $z \in \mathbb{C}_\pm$, see [MO], [KK]. The function $k(\cdot)$ is analytic on \mathcal{M} and satisfies $\sin k(z) = (1 - \Delta^2(z))^{\frac{1}{2}}$, $z \in \mathcal{M}$. Moreover, the quasimomentum $k(\cdot)$ is a conformal mapping from \mathcal{Z} onto the quasimomentum domain $\mathcal{K} = \mathbb{C} \setminus \bigcup \Gamma_n$, see Fig. 2 and 3. Here $\Gamma_n = (\pi n - ih_n, \pi n + ih_n)$ is a vertical slit with the height $h_n \geq 0$, $h_0 = 0$. The height h_n is defined by the equation $\cosh h_n = (-1)^n \Delta(e_n) \geq 1$, where $e_n \in [e_n^-, e_n^+]$ and $\Delta'(e_n) = 0$. The function $k(\cdot)$ maps the slit g_n^c on the slit Γ_n , and $k(-z) = -k(z)$ for all $z \in \mathcal{Z}$.

In order to describe the spectral properties of the operator H_0 we need the properties of ϑ, φ . Recall that ϑ, φ are the solutions of the equation $-y'' + py = z^2 y$ with the conditions $\varphi'(0, z) = \vartheta(0, z) = 1$ and $\varphi(0, z) = \vartheta'(0, z) = 0$. The solutions ϑ, φ satisfy the equations

$$\begin{aligned} \vartheta(x, z) &= \cos zx + \int_0^x \frac{\sin z(x-s)}{z} p(s) \vartheta(s, z) ds, \\ \varphi(x, z) &= \frac{\sin zx}{z} + \int_0^x \frac{\sin z(x-s)}{z} p(s) \varphi(s, z) ds. \end{aligned} \quad (2.4)$$

For each $x \in \mathbb{R}$ the functions $\vartheta(x, z), \varphi(x, z)$ are entire in $z \in \mathbb{C}$ and satisfy

$$\max \left\{ |z|_1 |\varphi(x, z)|, |\varphi'(x, z)|, |\vartheta(x, z)|, \frac{1}{|z|_1} |\vartheta'(x, z)| \right\} \leq X = e^{|\operatorname{Im} z| x + \|p\|_x},$$

$$\left| \varphi(x, z) - \frac{\sin zx}{z} \right| \leq \frac{X}{|z|^2} \|p\|_x, \quad |\vartheta(x, z) - \cos zx| \leq \frac{X}{|z|} \|p\|_x, \quad |z|_1 = \max\{1, |z|\}, \quad (2.5)$$

where $\|p\|_t = \int_0^t |p(s)| ds$ and $(x, z) \in \mathbb{R} \times \mathbb{C}$, see [PT]. These estimates yield

$$\beta(z) = \frac{\varphi'(1, z) - \vartheta(1, z)}{2} = \int_0^1 \frac{\sin z(2x-1)}{z} p(x) dx + \frac{O(e^{|\operatorname{Im} z|})}{z^2} \quad \text{as } |z| \rightarrow \infty. \quad (2.6)$$

Moreover, if $z = \pi n + O(1/n)$, then we obtain

$$\beta(z) = (-1)^n \frac{p_{sn} + O(n^{-1})}{2\pi n}, \quad p_{sn} = \int_0^1 p(x) \sin 2\pi n x dx. \quad (2.7)$$

The Floquet solutions $\psi^\pm(x, z), z \in \mathcal{Z}$ of the equation $-y'' + py = z^2 y$ are given by

$$\psi^\pm(x, z) = \vartheta(x, z) + m_\pm(z) \varphi(x, z), \quad m_\pm = \frac{\beta \pm i \sin k}{\varphi(1, \cdot)}, \quad (2.8)$$

where $\varphi(1, z) \psi^\pm(\cdot, z) \in L^2(\mathbb{R}_+)$ for all $z \in \mathbb{C}_+ \cup \cup g_n$. If $p = 0$, then $k = z$ and $\psi^\pm(x, z) = e^{\pm izx}$.

Let $\mathcal{D}_r(z_0) = \{|z - z_0| < r\}$ be a disk for some $r > 0, z_0 \in \Lambda$. It is well known that if $g_n = \emptyset$ for some $n \in \mathbb{Z}$, then the functions $\sin k(\cdot)$ and m_\pm **are analytic in some disk** $\mathcal{D}_\varepsilon(\mu_n) \subset \mathcal{Z}, \varepsilon > 0$ and the functions $\sin k(z)$ and $\varphi(1, z)$ have the simple zero at μ_n , see [F1]. Moreover, m_\pm satisfies

$$m_\pm(\mu_n) = \frac{\beta'(\mu_n) \pm i(-1)^n k'(\mu_n)}{\partial_z \varphi(1, \mu_n)}, \quad \operatorname{Im} m_\pm(\mu_n) \neq 0. \quad (2.9)$$

Furthermore, $\operatorname{Im} m^+(z) > 0$ for all $(z, n) \in (z_{n-1}^+, z_n^-) \times \mathbb{N}$ and the asymptotics hold true:

$$m_\pm(z) = \pm iz + O(1) \quad \text{as } |z| \rightarrow \infty, \quad z \in \mathcal{Z}_\varepsilon$$

where $\mathcal{Z}_\varepsilon = \{z \in \mathcal{Z} : \operatorname{dist}\{z, g_n\} > \varepsilon, g_n \neq \emptyset, n \in \mathbb{Z}\}, \varepsilon > 0. \quad (2.10)$

The function $\sin k$ and each function $\varphi(1, \cdot) \psi^\pm(x, \cdot), x \in \mathbb{R}$ are analytic on the Riemann surface \mathcal{M} . Recall that the Floquet solutions $\psi_\pm(x, z), (x, z) \in \mathbb{R} \times \mathcal{M}$ satisfy (see [T])

$$\psi_\pm(0, z) = 1, \quad \psi^\pm(0, z)' = m_\pm(z), \quad \psi^\pm(1, z) = e^{\pm ik(z)}, \quad \psi^\pm(1, z)' = e^{\pm ik(z)} m_\pm(z), \quad (2.11)$$

$$\psi^\pm(x, z) = e^{\pm ik(z)x} (1 + O(1/z)) \quad . \quad (2.12)$$

as $|z| \rightarrow \infty, z \in \mathcal{Z}_\varepsilon$, uniformly in $x \in \mathbb{R}$. Below we need the simple identities

$$\beta^2 + 1 - \Delta^2 = 1 - \varphi'(1, \cdot) \vartheta(1, \cdot) = -\varphi(1, \cdot) \vartheta'(1, \cdot). \quad (2.13)$$

Introduce the fundamental solutions $\Psi^\pm(x, z)$ of the equation

$$-\Psi^{\pm''} + (p + q)\Psi^\pm = z^2 \Psi^\pm, x \geq 0, \quad \Psi^\pm(x, z) = \psi^\pm(x, z), \quad x \geq t, \quad z \in \mathcal{Z} \setminus \{0\}. \quad (2.14)$$

Each function $\varphi(1, z) \Psi^\pm(x, z), x \geq 0$ is analytic in \mathcal{M} , since each $\varphi(1, z) \psi^\pm(x, z), x \geq 0$ is analytic in \mathcal{M} . We define **the modified Jost function** $\Psi_0^\pm = \Psi^\pm(0, z)$, which is meromorphic

in \mathcal{M} and has branch points $e_n^\pm, g_n \neq \emptyset$. The kernel of the resolvent $R = (H - z^2)^{-1}, z \in \mathbb{C}_+$, has the form

$$R(x, x', z) = \frac{\Phi(x, z)\Psi^+(x', z)}{\Psi_0^+(z)}, \quad x < x', \quad \text{and} \quad R(x, x', z) = R(x', x, z), \quad x > x'. \quad (2.15)$$

Recall that $\Phi(x, z)$ is the solution of the equation $-\Phi'' + (p + q)\Phi = z^2\Phi, x \geq 0, \Phi(0, z) = 0, \Phi'(0, z) = 1, z \in \mathbb{C}$. Each function $R(x, x', z), x, x' \in \mathbb{R}$ is meromorphic in \mathcal{M} . Remark that if $z_0 \in g_n^\pm \setminus \{\mu_n \pm i0\}$ and $\Psi_0^+(z_0) \neq 0$ for some n , then the resolvent of H is analytic at z_0 . The function $\Psi_0^+(z)$ has finite number of simple zeros on each $g_n^+, n \neq 0$ and on $i\mathbb{R}_+$ (no zeros on $\mathbb{C}_+ \setminus i\mathbb{R}_+$), where the squared zero is an eigenvalue. A pole of $\mathcal{R}(x, z) = \Psi^+(x, z)/\Psi_0^+(z)$ on g_n^+ is called a bound state. Of course, z^2 is really the energy, but since the momentum z is the natural parameter, we will abuse the terminology. Moreover, $\Psi_0^+(z)$ has infinite number of zeros in \mathbb{C}_- , see (1.15). We rewrite Definition A about the resonances on the Riemann surface Λ in the equivalent form in terms of the resonances on the Riemann surface \mathcal{M} .

Definition M. 1) A point $\zeta \in \overline{\mathbb{C}}_- \cap \mathcal{M}, \zeta \neq 0$ is a resonance, if the function $\mathcal{R}(x, z)$ has a pole at ζ for almost all $x > 0$.

2) A point $\zeta = e_n^\pm, n \neq 0$ (or $\zeta = 0$) is a virtual state, if the function $\mathcal{R}(x, \zeta + z^2)$ (or $\mathcal{R}(x, z)$) has a pole at ζ for almost all $x > 0$.

3) A point $\zeta \in \mathcal{M}$ is a state, if it is a bound state or a resonance or a virtual state. If $\zeta \in g_n^-, n \neq 0$ or $\zeta \in g_0^- = i\mathbb{R}_-$, then we call ζ an antibound state.

Let $\sigma_{bs}(H)$ (or $\sigma_{rs}(H)$ or $\sigma_{vs}(H)$) be the set of all bound states in the momentum variable $z = \sqrt{\lambda} \in \mathcal{M}$ (or resonances or virtual states) of H and let $\sigma_{st}(H) = \sigma_{vs}(H) \cup \sigma_{rs}(H) \cup \sigma_{bs}(H)$.

The kernel of the resolvent $R_0(z) = (H_0 - z^2)^{-1}, z \in \mathbb{C}_+$ has the form

$$R_0(x, x', z) = \varphi(x, z)\psi^+(x', z), \quad x < x', \quad \text{and} \quad R_0(x, x', z) = R_0(x', x, z), \quad x > x'. \quad (2.16)$$

Consider the states $z_n^0 = \sqrt{\lambda_n^0} \in g_n^c$ of H_0 . Due to (2.8), the function $\Psi_0^+ = \psi_0^+(0, \cdot) = 1$ and $\mathcal{R}(x, \cdot) = \vartheta(x, \cdot) + m_+\varphi(x, \cdot)$. Recall that $\varphi(1, \cdot)m_+$ and $\sin k(z)$ are analytic in \mathcal{M} . Thus the resolvent $R_0(z)$ has singularities only at $\mu_n \pm i0$, where $g_n \neq \emptyset$, and in order to describe the states of H_0 we need to study m_+ on g_n^c only. Let $\mathcal{A}(z_0), z_0 \in \mathcal{M}$ be the set of analytic functions in some disk $\mathcal{D}_r(z_0) = \{|z - z_0| < r\}, r > 0$. We need the following result (see [Zh3])

Lemma 2.1. All states of H are given by $\mu_n \pm i0 \in g_n^c, n \neq 0$, where $g_n \neq \emptyset$. Let the momentum gap $g_n = (e_n^-, e_n^+) \neq \emptyset$ for some $n \geq 1$. Then

i) $z_n^0 = \mu_n + i0 \in g_n^+$ is a bound state of H_0 iff one condition from (1)-(3) holds true

$$\begin{aligned} (1) \quad & m_- \in \mathcal{A}(\mu_n + i0), \\ (2) \quad & \beta(\mu_n) = i \sin k(\mu_n + i0) = -(-1)^n \sinh h_{sn}, \quad k(\mu_n + i0) = \pi n + i h_{sn} \quad h_{sn} > 0 \\ (3) \quad & m_+(z_n^0 + z) = \frac{c_n + O(z)}{z} \quad \text{as } z \rightarrow 0, \quad z \in \mathbb{C}_+, \quad c_n = \frac{-2 \sinh |h_{sn}|}{(-1)^n \partial_z \varphi(1, \mu_n)} < 0. \end{aligned} \quad (2.17)$$

ii) $z_n^0 = \mu_n - i0 \in g_n^-$ is an antibound state of H_0 iff one condition from (1)-(3) holds true

$$\begin{aligned} (1) \quad & m_- \in \mathcal{A}(\mu_n - i0), \\ (2) \quad & \beta(\mu_n) = i \sin k(\mu_n - i0) = -(-1)^n \sinh h_{sn}, \quad k(\mu_n - i0) = \pi n + i h_{sn}, \quad h_{sn} < 0, \\ (3) \quad & m_+(z_n^0 + z) = \frac{-c_n + O(z)}{z} \quad \text{as } z \rightarrow 0, \quad z \in \mathbb{C}_-. \end{aligned} \quad (2.18)$$

iii) $z_n^0 = \mu_n$ is a virtual state of H_0 iff one condition from (1)-(2) holds true

$$(1) \quad z_n^0 = \mu_n = e_n^- \quad \text{or} \quad z_n^0 = \mu_n = e_n^+,$$

$$(2) \quad m_+(z_n^0 + z) = \frac{c_n^0 + O(z)}{\sqrt{z}} \quad \text{as } z \rightarrow 0, \quad z \in \mathbb{C}_+, \quad c_n^0 \neq 0. \quad (2.19)$$

These simple facts are well known in the inverse spectral theory, see [N-Z], [MO] or [K5]. Remark that detail analysis of H_0 was done in [Zh3].

If $\mu_n \in g_n \neq \emptyset$, then the function m_+ has a pole at $z_n^0 = \mu_n + i0 \in g_n^+$ (a bound state) or at $z_n^0 = \mu_n - i0 \in g_n^-$ (an antibound state). If $\mu_n = e_n^+$ (or $\mu_n = e_n^-$), then $z_n^0 = \mu_n$ is a virtual state. Note that if some $g_n = \emptyset, n \neq 0$, then each $\psi^\pm(x, \cdot) \in \mathcal{A}(\mu_n), x \geq 0$. Moreover, the resolvent $R_0(z)$ has a pole at z_0 iff the function $m_+(\cdot)$ has a pole at z_0 .

The following asymptotics from [MO] hold true as $n \rightarrow \infty$:

$$\mu_n = \pi n + \varepsilon_n(p_0 - p_{cn} + O(\varepsilon_n)), \quad p_{cn} = \int_0^1 p(x) \cos 2\pi n x dx, \quad \varepsilon_n = \frac{1}{2\pi n}, \quad (2.20)$$

$$h_{sn} = -\varepsilon_n(p_{sn} + O(\varepsilon_n)), \quad (2.21)$$

$$e_n^\pm = \pi n + \varepsilon_n(p_0 \pm |p_n| + O(\varepsilon_n)), \quad p_n = \int_0^1 p(x) e^{-i2\pi n x} dx = p_{cn} - ip_{sn}. \quad (2.22)$$

Let $\varphi(x, z, \tau), (z, \tau) \in \mathbb{C} \times \mathbb{R}$ be the solutions of the equation

$$-\varphi'' + p(x + \tau)\varphi = z^2\varphi, \quad \varphi(0, z, \tau) = 0, \quad \varphi'(0, z, \tau) = 1. \quad (2.23)$$

Let y_1, y_2 be the solutions of the equations $-y'' + (p + q)y = z^2y, z \in \mathbb{C}$ and satisfying

$$y_2'(t, z) = y_1(t, z) = 1, \quad y_2(t, z) = y_1'(t, z) = 0. \quad (2.24)$$

Thus, they satisfy the integral equation

$$y_1(x, z) = \cos z(x - t) - \int_x^t \frac{\sin z(x - \tau)}{z} (p(\tau) + q(\tau)) y_1(\tau, z) d\tau,$$

$$y_2(x, z) = \frac{\sin z(x - t)}{z} - \int_x^t \frac{\sin z(x - \tau)}{z} (p(\tau) + q(\tau)) y_2(\tau, z) d\tau. \quad (2.25)$$

For each $x \in \mathbb{R}$ the functions $y_1(x, z), y_2(x, z)$ are entire in $z \in \mathbb{C}$ and satisfy

$$\max\{|z|_1 y_2(x, z)|, |y_2'(x, z)|, |y_1(x, z)|, \frac{1}{|z|_1} |y_1'(x, z)|\} \leq X_1 = e^{|\operatorname{Im} z| |t-x| + \int_x^t |p+q| ds},$$

$$|y_1(x, z) - \cos z(x - t)| \leq \frac{X_1}{|z|} \|q\|_t, \quad \left| y_2(x, z) - \frac{\sin z(x - t)}{z} \right| \leq \frac{X_1}{|z|^2} \|q\|_t, \quad (2.26)$$

and recall that $|z|_1 = \max\{1, |z|\}$ and $\|p\|_t = \int_0^t |p(s)| ds$.

The equation $-y'' + (p - z^2)y = f, y(0) = y'(0) = 0$ has an unique solution given by $y = \int_0^x \varphi(x - \tau, z, \tau) f(\tau) d\tau$. Hence the solutions Φ and Ψ_\pm of the $-y'' + (p + q)y = z^2y$ satisfy

$$\Phi(x, z) = \varphi(x, z) + \int_0^x \varphi(x - s, z, s) q(s) \Phi(s, z) ds, \quad (2.27)$$

$$\Psi^\pm(x, z) = \psi^\pm(x, z) - \int_x^t \varphi(x - s, z, s) q(s) \Psi^\pm(s, z) ds. \quad (2.28)$$

Below we need the well known fact for scattering theory

$$\Psi^+(0, z) = D(z^2) = \det(I + q(H_0 - z^2)^{-1}), \quad z \in \mathcal{M}. \quad (2.29)$$

It is similar to the case $p = 0$, see [J]. The case on the real with $p \neq \text{const}$ was considered in [F4]. The functions Ψ^\pm, m_\pm, \dots are meromorphic in \mathcal{M} and real on $i\mathbb{R}$. Then the following identities hold true:

$$\Psi^\pm(-z) = \overline{\Psi^\pm(\bar{z})}, \quad m_\pm(-z) = \overline{m_\pm(\bar{z})}, \quad \dots, \quad z \in \mathcal{Z}. \quad (2.30)$$

Lemma 2.2. *i) The following identities and asymptotics hold true:*

$$\Psi^\pm = \psi^\pm(t, \cdot) y_1 + \dot{\psi}^\pm(t, \cdot) y_2, \quad \text{where} \quad \dot{u} = \partial_t u, \quad (2.31)$$

$$\Psi^\pm(0, z) = 1 + \int_0^t \varphi(x, z) q(x) \Psi^\pm(x, z) dx, \quad (2.32)$$

$$\Psi^\pm(x, z) = e^{\pm ik(z)x} (1 + e^{\pm(t-x)(|v|-v)} O(1/z)), \quad v = \text{Im } z, \quad (2.33)$$

as $|z| \rightarrow \infty, z \in \mathcal{Z}_\varepsilon, \varepsilon > 0$ uniformly in $x \in [0, t]$. Moreover, (1.6) hold true.

ii) The function $\Psi^\pm(0, \cdot)$ has exponential type $2t$ in the half plane \mathbb{C}_\mp .

Proof. i) Using (2.24), (2.14) we obtain (2.31).

The identity $\varphi(x, \cdot, \tau) = \vartheta(\tau, z) \varphi(x + \tau, z) - \varphi(\tau, z) \vartheta(x + \tau, z)$ gives $\varphi(-x, \cdot, x) = -\varphi(x, z)$ and (2.28) yield (2.32).

Substituting (2.12) into (2.28) we obtain (2.33). In particular, substitution of (2.33), (2.26) into (2.32) yields (1.6), since $D(z^2) = \Psi^+(0, z)$.

ii) We give the proof for the case $\Psi^+(0, z)$, the proof for $\Psi^-(0, z)$ is similar. Due to (2.33), $\Psi^+(0, z)$ has exponential type $\leq 2t$ in the half plane \mathbb{C}_- . The decompositions $f(x, z) \equiv e^{-ixk(z)} \Psi^+(x, z) = 1 + \varepsilon f_1(x, z)$ and $\varphi(x, z) e^{ixk(z)} \equiv \varepsilon (e^{i2xz} - 1 + \varepsilon \eta(x, z))$, $\varepsilon = \frac{1}{2iz}$ give

$$\begin{aligned} \Psi^+(0, z) - 1 &= \int_0^t \varphi(x, z) e^{ixk} q(x) f(x, z) dx \\ &= \varepsilon \int_0^t e^{i2xz} q(x) f(x, z) (1 + \varepsilon e^{-i2xz} \eta(x, z)) dx - \varepsilon \int_0^t q(x) f(x, z) dx = \varepsilon K(z) - \varepsilon \int_0^t q(x) dx, \\ K &= \varepsilon \int_0^t e^{i2xz} q(x) (1 + G(x, z)) dx, \quad G = \varepsilon f_1 + \varepsilon e^{-i2xz} (\eta f - f_1), \quad z \in \mathcal{Z}_\varepsilon. \end{aligned} \quad (2.34)$$

Asymptotics (2.5), (2.33) and $k(z) = z + O(1/z)$ as $z \rightarrow \infty$ (see [KK]) yield

$$\eta(x, z) = e^{2x|\text{Im } z|} O(1), \quad f_1(x, z) = e^{2(t-x)|\text{Im } z|} O(1) \quad \text{as } |z| \rightarrow \infty, z \in \mathcal{Z}_\varepsilon. \quad (2.35)$$

We need the following variant of the Paley Wiener Theorem from [Fr]:

let $h \in \mathcal{Q}_t$ and let each $F(x, z), x \in [0, t]$ be analytic for $z \in \mathbb{C}_-$ and $F \in L^2((0, t) dx, \mathbb{R} dz)$. Then $\int_0^t e^{2izx} h(x) (1 + F(x, z)) dx$ has exponential type at least $2t$ in \mathbb{C}_- .

We can not apply this result to the function $K(z), z \in \mathbb{C}_-$, since $\psi^+(x, z)$ has a singularity at z_n^0 if $g_n \neq \emptyset$. But we can use this result for the function $K(z - i), z \in \mathbb{C}_-$, since (2.34), (2.35) imply $\sup_{x \in [0, 1]} |G(x, -i + \tau)| = O(1/\tau)$ as $\tau \rightarrow \pm\infty$. Then the function $\Psi^+(0, z)$ has exponential type $2t$ in the half plane \mathbb{C}_- . ■

3. SPECTRAL PROPERTIES OF H

Recall that an entire function $f(z)$ is said to be of *exponential type* if there is a constant A such that $|f(z)| \leq \text{const. } e^{A|z|}$ everywhere (see [Koo]). The infimum over the set of A for which such an inequality holds is called the type of f . The function f is said to belong to the Cartwright class Cart_ω if $f(z)$ is entire, of exponential type, $\omega_\pm(f) = \omega > 0$, where $\omega_\pm(f) = \limsup_{y \rightarrow \infty} \frac{\log |f(\pm iy)|}{y}$ and $\int_{\mathbb{R}} \frac{\log^+ |f(x)| dx}{1+x^2} < \infty$.

Let for shortness

$$\varphi_1 = \varphi(1, z), \quad \varphi'_1 = \varphi'(1, z), \quad \vartheta_1 = \vartheta(1, z), \dots, \Phi_1 = \Phi(1, z), \quad \Phi'_1 = \Phi'(1, z).$$

Define the functions

$$F(z) = \varphi(1, z)\Psi^-(0, z)\Psi^+(0, z), \quad z \in \mathcal{Z}, \quad (3.1)$$

which is real on \mathbb{R} , since $\Psi^-(0, z) = \overline{\Psi^+(0, \bar{z})}$ for all $z \in \mathcal{Z}$, see also (3.2).

Lemma 3.1. *i) The following identities and estimates hold true:*

$$F = \varphi(1, \cdot, t)y_1^2(0, \cdot) + \dot{\varphi}(1, \cdot, t)y_1(0, \cdot)y_2(0, \cdot) - \vartheta'(1, \cdot, t)y_2^2(0, \cdot) \in \text{Cart}_{1+2t}, \quad (3.2)$$

$$\varphi_1 \dot{\psi}_t^+ \dot{\psi}_t^- = -\vartheta'(1, \cdot, t), \quad \varphi_1(\dot{\psi}_t^+ \psi_t^- + \psi_t^+ \dot{\psi}_t^-) = \dot{\varphi}(1, \cdot, t) = \varphi'(1, \cdot, t) - \vartheta(1, \cdot, t), \quad (3.3)$$

$$\left| F(z) - \frac{\sin z}{z} \right| \leq \frac{C_F e^{(1+2t)|\text{Im } z|}}{|z|^2}, \quad C_F = 3(\|p\|_1 + \|p+q\|_t) e^{2\|p+q\|_t + \|p\|_1}. \quad (3.4)$$

ii) The set of zeros of F is symmetric with respect to the real line and the imaginary line. In each disk $\{z : |z - \pi n| < \frac{\pi}{4}\}$, $|n| \geq 1 + 4C_F e^{t\frac{\pi}{2}}$ there exists exactly one simple real zero z_n of F and F has not zeros in the domain $\mathcal{D}_F \cap \mathbb{C}_-$, where $\mathcal{D}_F = \{z \in \mathbb{C} : 4C_F e^{2|\text{Im } z|} < |z|\}$.

iii) For all $z \in \mathcal{Z}$ the following identity holds true:

$$\Psi_0^\pm(z) = e^{\pm ik(z)n_t} w_\pm(z), \quad w_\pm(z) = \Phi'(n_t, z) - m_\pm(z)\Phi(n_t, z), \quad n_t = \inf_{n \in \mathbb{N}, n \geq t} n. \quad (3.5)$$

Proof. i) The function $\varphi(1, z, t)$ for all $(t, z) \in R \times \mathbb{C}$ satisfies the following identity

$$\varphi(1, \cdot, t) = -\vartheta'_1 \varphi_t^2 + \varphi_1 \vartheta_t^2 + 2\beta \varphi_t \vartheta_t = \varphi_1 \psi_t^+ \psi_t^-, \quad (3.6)$$

see [Tr]. Recall that $\varphi(1, z, t), \vartheta(1, z, t)$ are define in (2.23). Using (3.6) we obtain

$$\begin{aligned} \dot{\varphi}(1, \cdot, t) &= \varphi_1(\dot{\psi}_t^+ \psi_t^- + \psi_t^+ \dot{\psi}_t^-), \\ \ddot{\varphi}(1, \cdot, t) &= \varphi_1(\ddot{\psi}_t^+ \psi_t^- + \psi_t^+ \ddot{\psi}_t^- + 2\dot{\psi}_t^+ \dot{\psi}_t^-) = 2\varphi_1(p(t) - z^2)\psi_t^+ \psi_t^- + 2\varphi_1 \dot{\psi}_t^+ \dot{\psi}_t^-. \end{aligned} \quad (3.7)$$

Identity (2.31) gives

$$F = \varphi_1(\psi_t^+ \psi_t^- y_1^2(0, \cdot) + \dot{\psi}_t^+ \dot{\psi}_t^- y_2^2(0, \cdot) + (\psi_t^+ \dot{\psi}_t^- + \dot{\psi}_t^+ \psi_t^-) y_1(0, \cdot) y_2(0, \cdot)). \quad (3.8)$$

Then using the following identities from [IM]

$$\dot{\varphi}(1, z, t) = \varphi'(1, z, t) - \vartheta(1, z, t), \quad \ddot{\varphi}(1, z, t) = 2(p(t) - z^2)\varphi(1, z, t) - 2\vartheta'(1, z, t), \quad (3.9)$$

and (3.6), (3.8) we obtain (3.2), (3.3), since Lemma 2.2, ii) and (2.5) yields $F \in \text{Cart}_{1+2t}$.

We will show (3.4). We have

$$y_1(0, \cdot) = \cos tz + \tilde{y}_1, \quad \tilde{y}_1 = \int_0^t \frac{\sin zs}{z} (p(s) + q(s)) y_1(s, \cdot) ds, \quad (3.10)$$

$$y_2(0, \cdot) = -\frac{\sin tz}{z} + \tilde{y}_2, \quad \tilde{y}_2 = \int_0^t \frac{\sin zs}{z} (p(s) + q(s)) y_2(s, \cdot) ds, \quad (3.11)$$

$$\vartheta(1, t) = \cos z + \vartheta_{1t}, \quad \vartheta_{1t} = \int_0^1 \frac{\sin z(1-s)}{z} p(s+\tau) \vartheta(s, \tau) ds, \quad (3.12)$$

$$\vartheta'(1, t) = -z \sin z + \vartheta'_{1t}, \quad \vartheta'_{1t} = \int_0^1 \cos z(1-s) p(s+t) \vartheta(s, t) ds, \quad (3.13)$$

$$\varphi(1, t) = \frac{\sin z}{z} + \varphi_{1t}, \quad \varphi_{1t} = \int_0^1 \frac{\sin z(1-s)}{z} p(s+t) \varphi(s, t) ds, \quad (3.14)$$

Then (3.2) imply

$$\begin{aligned} F &= (\cos tz + \tilde{y}_1)^2 \left(\frac{\sin z}{z} + \varphi_{1t} \right) + \left(\frac{\sin tz}{z} - \tilde{y}_2 \right)^2 (z \sin z - \vartheta'_{1t}) + (\cos tz + \tilde{y}_1) \left(-\frac{\sin tz}{z} + \tilde{y}_2 \right) \dot{\varphi}(1, z, t) \\ &= \frac{\sin tz}{z} + f_1 + f_2 + f_3, \end{aligned}$$

where

$$\begin{aligned} f_1 &= y_1(0, \cdot)^2 \varphi_{1t} + \frac{\sin tz}{z} (\cos tz + y_1(0, \cdot)) \tilde{y}_1, \\ f_2 &= -y_2(0, \cdot)^2 \vartheta'_{1t} + z \sin z (y_2(0, \cdot) - \frac{\sin tz}{z}) \tilde{y}_2, \quad f_3 = y_1(0, \cdot) y_2(0, \cdot) \dot{\varphi}(1, z, t), \end{aligned}$$

$$|f_3| \leq \frac{C_t}{|z|^2} \|p\|_1, \quad |f_j| \leq \frac{C_t}{|z|^2} (\|p\|_1 + 2\|p+q\|_t), \quad j = 1, 2,$$

which yields (3.4), where $\|q\|_t = \int_0^t |q(x)| dx$ and $C_t = e^{(2t+1)|\operatorname{Im} z| + 2\|p+q\|_t + \|p\|_1}$.

ii) Using (3.4) we obtain for $|n| \geq 1 + 4C_F e^{t\frac{\pi}{2}}$

$$\left| F(z) - \frac{\sin z}{z} \right| \leq \frac{C_F}{|z|^2} e^{|\operatorname{Im} z| + t\frac{\pi}{2}} \leq \frac{4C_F}{|z|} e^{t\frac{\pi}{2}} \frac{|\sin z|}{|z|} < \frac{|\sin z|}{|z|} \quad \text{all } |z - \pi n| = \frac{\pi}{4},$$

since $e^{|\operatorname{Im} z|} \leq 4|\sin z|$ for all $|z - \pi n| \geq \pi/4, n \in \mathbb{Z}$, (see [PT]). Hence, by Rouché's theorem, F has as many roots, counted with multiplicities, as $\sin z$ in each disk $\mathcal{D}_{\frac{\pi}{4}}(\pi n)$. Since $\sin z$ has only the roots $\pi n, n \geq 1$, and i) of the lemma follows. This zero in $\mathcal{D}_{\frac{\pi}{4}}(\pi n)$ is real since F is real on the real line.

Using (3.4) and $e^{|\operatorname{Im} z|} \leq 4|\sin z|$ for all $|z - \pi n| \geq \pi/4, n \in \mathbb{Z}$, we obtain

$$|F(z)| \geq \left| \frac{\sin z}{z} \right| - \left| F(z) - \frac{\sin z}{z} \right| \geq \frac{e^{|\operatorname{Im} z|}}{4|z|^2} \left(|z| - 4C_F e^{2t|\operatorname{Im} z|} \right) > 0,$$

for all $z \in \mathcal{D}_1 = \{z \in \mathcal{D}_F : |z - \pi n| \geq \pi/4, n \in \mathbb{Z}\}$. This yields $|F| > 0$ in \mathcal{D}_1 . But the function F has exactly one real zero z_n in $\mathcal{D}_{\frac{\pi}{4}}(\pi n), n \geq n_0$. Then F has not zeros in the domain \mathcal{D}_F . The function F is real on the real line and the imaginary line. Hence the set of zeros of F is symmetric with respect to the real line and the imaginary line.

iii) Using $\Psi^\pm(0, z) = \{\Psi^\pm(x, z), \Phi(x, z)\}$, $z \in \mathcal{Z}$ at $x = n_t$, and (2.11) we obtain (3.5), where $\{y, u\} = yu' - y'u$ is the Wronskian. ■

Let $\vartheta, \tilde{\varphi}$ be the solutions of the equations $-y'' + (p+q)y = z^2 y, z \in \mathbb{C}$ and satisfying

$$\tilde{\vartheta}(x, z) = \vartheta(x, z), \quad \tilde{\varphi}(x, z) = \varphi(x, z), \quad \text{all } x \geq t.$$

Hence the solutions $\tilde{\vartheta}, \tilde{\varphi}$ satisfy the equations

$$\begin{aligned}\tilde{\vartheta}(x, z) &= \vartheta(x, z) - \int_x^t \varphi(x-s, z, s) q(s) \tilde{\vartheta}(s, z) ds, \\ \tilde{\varphi}(x, z) &= \varphi(x, z) - \int_x^t \varphi(x-s, z, s) q(s) \tilde{\varphi}(s, z) ds.\end{aligned}\quad (3.15)$$

For each $x \geq 0$ the functions $\tilde{\vartheta}(x, z), \tilde{\varphi}(x, z)$ are entire and real for $z^2 \in \mathbb{R}$. The identities (3.15) and (2.14) give

$$\Psi^\pm(x, z) = \tilde{\vartheta}(x, z) + m_\pm(z) \tilde{\varphi}(x, z), \quad \text{all } (x, z) \in \mathbb{R}_+ \times \mathcal{Z}. \quad (3.16)$$

Recall that $g_n^c = \overline{g}_n^- \cup \overline{g}_n^+$ and we define the sets

$$\mathcal{Z} = i\mathbb{R} \cup \mathbb{C}_- \cup \bigcup_{n \in \mathbb{Z}} g_n^c, \quad \mathcal{Z}_0 = \mathcal{Z} \setminus \{0, e_n^\pm, \mu_n \pm i0, g_n \neq \emptyset, n \in \mathbb{Z}\}. \quad (3.17)$$

Lemma 3.2. *i) If $g_n = (e_n^-, e_n^+) = \emptyset$ for some $n \neq 0$, then each $\Psi^\pm(x, \cdot) \in \mathcal{A}(\mu_n), x \geq 0$ and $\text{Im } \Psi^\pm(0, \mu_n) \neq 0$. Moreover, $\mu_n = e_n^\pm$ is a simple zero of F and $\mu_n \notin \sigma_{st}(H)$.*

ii) $\Psi^\pm(0, z) \neq 0$ for all $z \in (e_{n-1}^+, e_n^-), n \in \mathbb{Z}$. Moreover, states of H and zeros of $\Psi^+(0, z)$ belong to the set $\mathcal{Z} = \{i\mathbb{R}\} \cup \mathbb{C}_- \cup \bigcup g_n^c \subset \mathcal{Z}$.

iii) A point $z \in \mathcal{Z}_0$ is a zero of Ψ_0^+ iff $z \in \mathcal{Z}_0 \cap \sigma_{st}(H)$. In particular,

- 1) $z \in \overline{\mathbb{C}}_+ \cap \mathcal{Z}$ is a bound state of H ,
- 2) $z \in \overline{\mathbb{C}}_- \cap \mathcal{Z}$ is a resonance of H .

Proof. i) Lemma 2.1 and identity (3.16) yield that each $\Psi^\pm(x, \cdot) \in \mathcal{A}(\mu_n), x \geq 0$.

Using (2.9) we deduce that $\Psi_0^\pm(\mu_n) \neq 0$, then the functions $\Psi^\pm(x, \cdot), 1/\Psi_0^\pm \in \mathcal{A}(\mu_n)$. Thus μ_n is not a state of H and μ_n is a simple zero of F .

ii) The conformal mapping $k(\cdot)$ maps each interval $(e_{n-1}^+, e_n^-), n \geq 1$ onto $(\pi(n-1), \pi n)$. Recall that $\mu_n \in [e_n^-, e_n^+]$. Then the function m_\pm is analytic on (e_{n-1}^+, e_n^-) and $\text{Im } m_\pm(z) \neq 0$ for all $z \in (e_{n-1}^+, e_n^-)$. Then the identity (3.16) gives $\Psi^\pm(0, z) \neq 0$ for all $z \in (e_{n-1}^+, e_n^-)$.

Then the identity $\Psi^+(0, z) = D(z^2)$ and standard arguments (similar to the case $p = 0$, see [K1]) imply that states of H and zeros of $\Psi^+(0, z)$ belong to the set $\mathcal{Z} = \{i\mathbb{R}\} \cup \mathbb{C}_- \cup \bigcup g_n^c \subset \mathcal{Z}$.

iii) The statement iii) follows from the identities (3.16), (2.15). ■

We consider the properties of the states of H which coincide with unperturbed states z_n^0 .

Lemma 3.3. *Let $\zeta = \mu_n + i0 \in g_n^+$ or $\zeta = \mu_n - i0 \in g_n^-$ for some $n \geq 1$, where $g_n \neq \emptyset$. Then*

i) $\tilde{\varphi}(0, \mu_n) = 0$ iff $\Phi(n_t, \mu_n) = 0$, where $n_t = \inf_{n \in \mathbb{N}, n \geq t} n$.

ii) Let in addition $\zeta = z_n^0 \in \sigma_{st}(H_0)$. Then $\Psi_0^- \in \mathcal{A}(\zeta)$ and each $\Psi^+(x, \cdot), x > 0$ has a simple pole at ζ and there are two cases:

1) if $\tilde{\varphi}(0, \mu_n) = 0$, then $\Psi_0^+ \in \mathcal{A}(\zeta), \Psi_0^-(\zeta) \neq 0$ and $\zeta \in \sigma_{st}(H)$. In particular,

$$\text{if } \zeta = \mu_n + i0 \in g_n^+ \Rightarrow \Psi_0^+(\zeta) \neq 0, \quad F(\mu_n) = 0, \quad (-1)^n F'(\mu_n) > 0; \quad (3.18)$$

2) if $\tilde{\varphi}(0, \mu_n) \neq 0$, then

$$\begin{aligned}\frac{\Psi^+(x, \cdot)}{\Psi_0^+} &\in \mathcal{A}(\zeta), x \geq 0, \quad \Psi_0^-(\zeta) \neq 0, \quad \Psi_0^+(z) = \frac{c_n + O(\varepsilon)}{\varepsilon} \tilde{\varphi}(0), \\ &\zeta \notin \sigma_{st}(H), \quad F(\mu_n) \neq 0.\end{aligned}\quad (3.19)$$

iii) $\zeta \in \sigma_{bs}(H)$ (or $\zeta \in \sigma_{vs}(H)$) iff $\zeta \in \sigma_{bs}(H_0)$ (or $z_1 \in \sigma_{vs}(H_0)$) and $\Phi(n_t, \mu_n) = 0$.

iv) Let $\zeta \in \sigma_{st}(H_0) \cap \sigma_{st}(H)$, then the same number but on another sheet is not a state of H .

v) Let $\zeta \in \sigma_{st}(H_0)$ and let the same number $\zeta_1 = \bar{\zeta}$ but on another sheet is a state of H . Then $\zeta \notin \sigma_{st}(H)$.

Proof. i) Comparing (3.5) and (3.16) we deduce that $\tilde{\varphi}(0, \mu_n) = 0$ iff $\Phi(0, \mu_n) = 0$.

ii) Lemma 2.1 yields $m_- \in \mathcal{A}(\zeta)$ and each $\Psi^+(x, \cdot), x > 0$ has a simple pole at ζ and $m_+(z) = \frac{\pm c_n}{\varepsilon} + O(1)$ as $\varepsilon \rightarrow 0$, $\varepsilon \in \mathbb{C}_\pm$ and $c_n < 0$.

1) If $\tilde{\varphi}(0, \mu_n) = 0$, then (3.16) yields $\Psi_0^+ \in \mathcal{A}(\zeta)$, $\Psi_0^-(\zeta) \neq 0$. Thus $\zeta \in \sigma_{st}(H)$, since each $\Psi^+(x, \cdot), x > 0$ has a simple pole at ζ .

Consider the case $\zeta = \mu_n + i0 \in g_n^+$. Recall that (3.5) gives

$$\Psi^\pm(0, z) = e^{\pm ik(z)n_t} w_\pm(z), \quad w_\pm(z) = \Phi'(n_t, z) - m_\pm(z)\Phi(n_t, z), \quad n_t = \inf_{n \in \mathbb{N}, n \geq t} n, \quad z \in \mathcal{Z},$$

which yields

$$w_-(\zeta) = \Phi'(n_t, \mu_n) \neq 0, \quad w_+(z_1) = \Phi'(n_t, \mu_n) \left(1 - c_n \frac{\partial_z \Phi(n_t, \mu_n)}{\Phi'(n_t, \mu_n)}\right) \neq 0,$$

since $\Phi'(n_t, \mu_n) \neq 0$, $\Phi(n_t, \mu_n) = 0$ and $\Phi'(n_t, \mu_n) \partial_z \Phi(n_t, \mu_n) > 0$ (see [PT]). This yields (3.18), since $(-1)^n \partial_z \varphi(1, \mu_n) > 0$ (see [PT]).

2) Lemma 2.1 yields $m_+(z) = \frac{c_n + O(\varepsilon)}{\varepsilon}$ as $\varepsilon = z - \zeta \rightarrow 0$. Using (3.16), we obtain

$$\Psi_0^+(z) = \frac{c_n + O(\varepsilon)}{\varepsilon} \tilde{\varphi}(0), \quad \frac{\Psi^+(x, z)}{\Psi_0^+(z)} = \frac{\varepsilon \tilde{\vartheta}(x) - c_n \tilde{\varphi}(x) + O(\varepsilon)}{\varepsilon \tilde{\vartheta}(0) - c_n \tilde{\varphi}(0) + O(\varepsilon)} = \frac{\tilde{\varphi}(x)}{\tilde{\varphi}(0)} + O(\varepsilon) \quad \text{as } \varepsilon \rightarrow 0,$$

since $c_n \tilde{\varphi}(0, \mu_n) \neq 0$, where $\tilde{\varphi}(x) = \tilde{\varphi}(x, z), \dots$. This yields $\varphi(1, z) \Psi_0^+(z) = \partial_z \varphi(1, \mu_n) c_n + o(1)$ and $m_- \in \mathcal{A}(\zeta)$ gives $\Psi_0^+(\zeta) = \tilde{\vartheta}(0) \neq 0$, which yields $F(\mu_n) \neq 0$ and (3.19).

Using i) and ii) we obtain iii).

iv) $\Psi_0^+ \in \mathcal{A}(\zeta_1)$ and each $\Psi^+(x, \cdot) \in \mathcal{A}(\zeta_1), x > 0$. Due to ii) $\tilde{\varphi}(0, \mu_m) = 0$, then we obtain $\Psi_0^+(\zeta_1) \neq 0$. Thus $\zeta_1 \notin \sigma_{st}(H)$.

v) Assume that $\zeta \in \sigma_{st}(H)$. Then iv) gives contradiction. Thus $\zeta \notin \sigma_{st}(H)$. ■

Consider virtual states, which coincide with the points e_n^\pm .

Lemma 3.4. Let $\zeta = e_n^-$ or $\zeta = e_n^+$ for some $n \geq 1$, where $e_n^- < e_n^+$ and let $\varepsilon = z - \zeta$.

i) Let $\zeta \neq \mu_n$ and let $\Psi_0^+(\zeta) = 0$. Then ζ is a simple zero of F , $\zeta \in \sigma_{vs}(H)$ and

$$\Psi_0^+(z) = \tilde{\varphi}(0, \zeta) c \sqrt{\varepsilon} + O(\varepsilon), \quad \mathcal{R}(x, z) = \frac{\Psi^+(x, z) + O(\varepsilon)}{\tilde{\varphi}(0, \zeta) c \sqrt{\varepsilon}}, \quad c \tilde{\varphi}(0, \zeta) \neq 0. \quad (3.20)$$

ii) Let $\zeta = \mu_n$ and $\tilde{\varphi}(0, \zeta) \neq 0$. Then $F(\zeta) \neq 0$ and each $\mathcal{R}(x, \cdot), x > 0$ has not singularity at ζ , and $\zeta \notin \sigma_{vs}(H)$.

iii) Let $\zeta = \mu_n$ and $\tilde{\varphi}(0, \zeta) = 0$. Then $\zeta \in \sigma_{vs}(H)$, $\Psi_0^\pm(\zeta) \neq 0$ and ζ is a simple zero of F and each $\mathcal{R}^2(x, \cdot), x > 0$ has a pole at ζ .

Proof. i) Lemma 2.1 gives $m_\pm(z) = m_\pm(\zeta) + c \sqrt{\varepsilon} + O(\varepsilon)$ as $\varepsilon = z - \zeta \rightarrow 0, c \neq 0$. We have two cases: 1) Firstly, let $\tilde{\varphi}(0, \zeta) \neq 0$. Then identity (3.16) implies (3.20).

2) Secondly, if $\tilde{\varphi}(0, \zeta) = 0$, then (3.16) implies $\Psi_0^+(\zeta) = \tilde{\vartheta}(0, \zeta) \neq 0$, which gives contradictions.

ii) If $\zeta = \mu_n$, then Lemma 2.1 gives $m_\pm(z) = \pm \frac{c}{\sqrt{\varepsilon}} + O(1), \varepsilon \rightarrow 0, c \neq 0$. Then (3.16) implies

$$\Psi_0^\pm(z) = \pm \frac{\tilde{\varphi}(0, \zeta) c}{\sqrt{\varepsilon}} + O(1), \quad \frac{\Psi^+(x, z)}{\Psi_0^+(z)} = \frac{\tilde{\vartheta}(x, z) + (\frac{c}{\sqrt{\varepsilon}} + O(1)) \tilde{\varphi}(x, z)}{\frac{\tilde{\varphi}(0, \zeta) c}{\sqrt{\varepsilon}} + O(1)} = \frac{1 + O(\sqrt{\varepsilon})}{\tilde{\varphi}(0, \zeta)}.$$

Thus the function $\mathcal{R}(x, \cdot)$, $x > 0$ has not singularity at ζ and $\zeta \notin \sigma_{vs}(H)$, $F(\zeta) \neq 0$.

iii) If $\tilde{\varphi}(0, \zeta) = 0$, then (3.16) gives $\Psi_0^+(\zeta) = \tilde{\vartheta}(0, \zeta) \neq 0$, since $\tilde{\vartheta}(0, \zeta) \neq 0$ and $\beta(\zeta) = 0$. Moreover, we obtain $\Psi^+(x, z) = \tilde{\vartheta}(x, z) + (\frac{c}{\sqrt{\varepsilon}} + O(1))\tilde{\varphi}(x, z)$, and the function $\mathcal{R}^2(x, \cdot)$, $x > 0$ has a pole at ζ , $\zeta \in \sigma_{vs}(H)$ and $F(\zeta) = 0$. ■

Lemma 3.5. *Let $\lambda \in \gamma_n$, $\lambda \neq \mu_n^2$ be an eigenvalue of H for some $n \geq 0$ and let $z = \sqrt{\lambda} \in i\mathbb{R}_+ \cup \bigcup_{n \geq 1} g_n^+$. Then*

$$C_\lambda = \int_0^\infty |\Psi^+(x, z)|^2 dx = -\frac{\Psi^{+'}(0, z)}{2z} \partial_z \Psi^+(0, z) > 0, \quad (3.21)$$

$$\frac{i2 \sin k(z)}{\varphi(1, z)} = \Psi^-(0, z_1) \Psi^{+'}(0, z) \neq 0, \quad i \sin k(z) = -(-1)^n \sinh h, \quad h > 0, \quad (3.22)$$

$$C_\lambda = \frac{(-1)^n F'(z) \sinh h}{z \varphi^2(1, z) \Psi^-(0, z)^2} > 0, \quad \frac{(-1)^n F'(z)}{z} > 0. \quad (3.23)$$

Proof. Using the identity $\{\frac{\partial}{\partial z} \Psi^+, \Psi^+\}' = 2z(\Psi^+)^2$ we obtain (3.21).

Using the Wronskian for the functions Ψ^+ , Ψ^- and (2.11) we obtain $\Psi^-(0, z) \Psi^{+'}(0, z) = m_+(z) - m_-(z)$, which yields (3.22), since $k(z) = \pi n + ih$ for some $h > 0$, see the definition of $k(\cdot)$ before (2.11). Then identities (3.21), (3.22) imply (3.23). ■

4. PROOF OF MAIN THEOREMS

Proof of Theorem 1.1. i) Asymptotics (1.6) were proved in Lemma 2.2.

ii) and iii) of Lemma 3.3 give (1.8) for the case of non-virtual states, i.e., $\neq e_n^\pm$.

Lemma 3.4 implies (1.8) for the case of virtual states.

Lemma 3.2 gives (1.7) for the case of non-virtual states. Lemma 3.4 implies (1.7) for the case of virtual states.

ii) Using ii) and iii) of Lemma 3.3 we obtain (1.9).

iii) Due to i) ζ is a zero of F , then (3.4) yields (1.10). Lemma 3.1 (ii) completes the proof of iii). ■

Proof of Theorem 1.2. i) Let $g_n \neq \emptyset$. The entire function $F = \varphi(1, \cdot) \Psi_0^+ \Psi_0^-$ has different sign on σ_n and σ_{n+1} , since $\Psi_0^+(z) \Psi_0^-(z) = |\Psi_0^+(z)|^2 > 0$ for z inside $\sigma_n \cup \sigma_{n+1}$ (see (2.31)) and $\varphi(1, \cdot)$ has one simple zero in each interval $[e_n^-, e_n^+]$. Then F has an odd number of zeros on $[e_n^-, e_n^+]$.

By Lemma 3.2-3.4, $\zeta \in g_n^c$ is a state of H iff $\zeta \in \bar{g}_n$ is a zero of F (according to the multiplicity). Then the number of states on g_n^c is odd.

Using Lemma 3.1 and 3.2 we deduce that there exists an exactly one simple state z_n in each interval $[e_n^-, e_n^+]$ for $g_n \neq \emptyset$ and for $n \geq 1$ large enough. Moreover, asymptotics $e_n^\pm = \pi n + \frac{p_0 + o(1)}{2\pi n}$, see (2.22) give

$$z_n = \pi n + \frac{p_0 + o(1)}{2\pi n}. \quad (4.1)$$

Using arguments proving (2.32) we obtain the identities

$$\tilde{\vartheta}(0, z) = 1 + \int_0^t \varphi(x, z) q(x) \tilde{\vartheta}(x, z) dx, \quad \tilde{\varphi}(0, z) = \int_0^t \varphi(x, z) q(x) \tilde{\varphi}(x, z) dx, \quad (4.2)$$

The standard iteration procedure and (4.1) give the asymptotics

$$\tilde{\vartheta}(0, z_n) = 1 + O(1/n), \quad , \quad (4.3)$$

$$\tilde{\varphi}(0, z_n) = \int_0^t \frac{\sin^2 z_n x}{z_n^2} q(x) dx + O(1/n^3) = \frac{q_0 - \hat{q}_{cn} + O(\frac{1}{n})}{2(\pi n)^2}, \quad (4.4)$$

where $\hat{q}_{cn} = \int_0^t q(x) \cos 2\pi n x dx$. Using (2.13) and $\Psi^\pm = \tilde{\vartheta} + m_\pm \tilde{\varphi}$, see (3.16), we obtain

$$F = F_1 + F_2 + F_3, \quad F_1 = \varphi(1, \cdot) \tilde{\vartheta}_0^2, \quad F_2 = 2\beta \tilde{\vartheta}_0 \tilde{\varphi}_0, \quad F_3 = -\vartheta'(1, \cdot) \tilde{\varphi}_0^2, \quad (4.5)$$

where for shortness $\tilde{\vartheta}_0 = \tilde{\vartheta}(0, z)$, $\tilde{\varphi}_0 = \tilde{\varphi}(0, z)$. Using estimates (4.3), (4.4), we obtain

$$\begin{aligned} F_1(z_n) &= \varphi(1, z_n)(1 + O(n^{-1})), & F_3(z_n) &= O(n^{-4}) \\ F_2(z_n) &= (-1)^n \frac{(p_{sn} + O(\frac{1}{n})) (q_0 - \hat{q}_{cn} + O(\frac{1}{n}))}{\pi n} = f_n + O(n^{-4}) \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (4.6)$$

where $f_n = (-1)^n \frac{p_{sn}(q_0 - \hat{q}_{cn})}{2(\pi n)^3}$. Combine these asymptotics and the identity $F(z_n) = 0$ we get

$$\varphi(1, z_n) = -F_2(z_n) + O(n^{-4}) = -f_n + O(n^{-4}). \quad (4.7)$$

Then, using $\varphi(1, z_n) = \partial_z \varphi(1, \mu_n) \delta_n + O(n^{-4})$, where $z_n = \mu_n + \delta_n$, we obtain

$$\partial_z \varphi(1, \mu_n) \delta_n = -f_n + O(n^{-4})$$

and the asymptotics $\partial_z \varphi(1, \mu_n) = \frac{(-1)^n + O(\frac{1}{n})}{(\pi n)}$ give

$$\delta_n = -\frac{f_n}{\partial_z \varphi(1, \mu_n)} + O(n^{-3}) = -\frac{(\hat{q}_0 - \hat{q}_{cn}) p_{sn} + O(\frac{1}{n})}{2(\pi n)^2},$$

which yields (1.13).

Denote by $\mathcal{N}^+(r, f)$ the number of zeros of f with real part ≥ 0 having modulus $\leq r$, and by $\mathcal{N}^-(r, f)$ the number of its zeros with real part < 0 having modulus $\leq r$, each zero being counted according to its multiplicity. We recall the well known result (see [Koo]).

Theorem (Levinson). *Let the entire function $f \in \mathcal{C}^\rho$. Then $\mathcal{N}^\pm(r, f) = \frac{r}{\pi}(\rho + o(1))$ as $r \rightarrow \infty$, and for each $\delta > 0$ the number of zeros of f with modulus $\leq r$ lying outside both of the two sectors $|\arg z|, |\arg z - \pi| < \delta$ is $o(r)$ for $r \rightarrow \infty$.*

Let $\mathcal{N}(r, f)$ be the total number of zeros of f with modulus $\leq r$. Denote by $\mathcal{N}_+(r, f)$ (or $\mathcal{N}_-(r, f)$) the number of zeros of f with imaginary part > 0 (or < 0) having modulus $\leq r$, each zero being counted according to its multiplicity.

Let $s_0 = 0$ and $\pm s_n > 0, n \in \mathbb{N}$ be all real zeros of F and let n_0 be the multiplicity of the zero $s_0 = 0$. Define the entire function $F_1 = z^{n_0} \lim_{r \rightarrow \infty} \prod_{0 < s_n \leq r} (1 - \frac{z^2}{s_n^2})$. The Levinson Theorem and Lemma 2.2 imply

$$\mathcal{N}(r, F) = \mathcal{N}(r, F_1) + \mathcal{N}(r, F/F_1) = 2r \frac{1 + 2t + o(1)}{\pi}, \quad \mathcal{N}(r, F_1) = 2r \frac{1 + o(1)}{\pi}, \quad (4.8)$$

$$\mathcal{N}_-(r, F) = \mathcal{N}_+(r, F) = \mathcal{N}_-(r, \Psi_0^+) - N_0, \quad (4.9)$$

as $r \rightarrow \infty$, where N_0 is the number of non-positive eigenvalues of H . Thus

$$2\mathcal{N}_-(r, F) = 2r \frac{2t + o(1)}{\pi}, \quad (4.10)$$

which yields (1.14).

ii) Using Lemma 3.5 we obtain the statements ii) and iii). ■

Proof of Theorem 1.3. Let $z = e_n^\pm$. Identity (3.5) and $k(e_n^\pm) = \pi n$ yield

$$\Psi_0^-(z) = \Psi_0^+(z) = (-1)^N w_+(z), \quad w_+(z) = \Phi'(n_t, z) - \frac{\beta(z)}{\varphi(1, z)} \Phi(n_t, z), \quad N = n_t n. \quad (4.11)$$

Estimates (2.5) and $e_n^\pm = \pi n + \varepsilon_n(p_0 \pm |p_n| + O(\varepsilon_n))$, $\varepsilon_n = \frac{1}{2\pi n}$ (see (2.22)) give

$$\Phi'(n_t, z) = (-1)^N + \frac{O(1)}{n}, \quad \Phi(n_t, z) = \frac{\sin n_t z}{\pi n} + \frac{O(1)}{n^2} = \frac{O(1)}{n^2}. \quad (4.12)$$

Using (2.26), we obtain

$$\begin{aligned} \sin e_n^\pm &= (-1)^n \sin \frac{\pm |p_n| + O(\frac{1}{n})}{2\pi n} = (-1)^n \frac{\pm |p_n| + O(\frac{1}{n})}{2\pi n}, \\ \varphi(1, e_n^\pm) &= \frac{\sin e_n^\pm}{\pi n} + \frac{(-1)^n p_{cn}}{2\pi^2 n^2} + \frac{O(1)}{n^3} = (-1)^n \frac{\pm |p_n| + p_{cn} + O(\frac{1}{n})}{2\pi^2 n^2}. \end{aligned}$$

Then the estimate $\sqrt{x^2 + y^2} - y \geq x$ for $y, x \geq 0$ gives $|p_n| \pm p_{cn} \geq |p_{sn}|$, which yields

$$\frac{\beta(e_n^\pm)}{\varphi(1, e_n^\pm)} = \pi n \frac{p_{sn} + O(\frac{1}{n})}{\pm |p_n| + p_{cn} + O(\frac{1}{n})} = O(\pi n), \quad (4.13)$$

since $|p_{sn}| \geq \frac{1}{n^\alpha}$. Combining (4.11)-(4.13) and (2.7), we obtain $\Psi_0^+(e_n^\pm) = 1 + o(1)$. The function $\Psi_0^+(z)$ is analytic on g_n^- and $\Psi_0^+(e_n^\pm) = 1 + o(1)$. Thus $\Psi_0^+(z)$ has not zeros on g_n^- , since by Theorem 1.2, the function F has exactly 1 zero on each $\bar{g}_n \neq \emptyset$ at large $n > 1$.

Let $\mu_n + i0 \in g_n^+$ be a bound state of H_0 for some n large enough. Then Lemma 2.1 implies $h_{sn} > 0$. Moreover, (2.21) gives $h_{sn} = -\frac{p_{sn} + O(\frac{1}{n})}{2\pi n}$ as $n \rightarrow \infty$. Thus $p_{sn} < -\frac{1}{n^\alpha}$ at large $n > 1$ and asymptotics (1.13) gives that the bound state $z_n > \mu_n$ if $q_0 > 0$ and $z_n < \mu_n$ if $q_0 < 0$. The proof of other cases is similar. ■

Proof of Theorem 1.4. i) Using the identities (2.32) and (3.16) we obtain

$$\Psi_0^+ = Y_1 + \frac{i \sin k}{\varphi_1} \tilde{\varphi}(z_n), \quad Y_1 = \tilde{v}(z_n) + \frac{\beta}{\varphi_1} \tilde{\varphi}(z_n). \quad (4.14)$$

Note that (2.13) gives $\beta(\mu_n) = 0$. Then asymptotics (1.13), (2.6), (2.5) imply

$$\frac{\beta(z_n)}{\varphi(1, z_n)} = \frac{\beta'(\mu_n) + O(\varepsilon_n^3)}{\partial_z \varphi(1, \mu_n) + O(\varepsilon_n^3)} = o(1) \quad \text{as } n \rightarrow \infty, \quad \varepsilon_n = \frac{1}{2\pi n} \quad (4.15)$$

where we used asymptotics $\partial_z \varphi(1, \mu_n) = \frac{(-1)^n + O(\varepsilon_n)}{(\pi n)}$ and $\beta'(\mu_n) = \frac{o(1)}{n}$. Thus (4.3), (4.4) give

$$Y_1 = 1 + O(\varepsilon_n), \quad \tilde{\varphi}(0, z_n) = 2\varepsilon_n^2(b_n + O(\varepsilon_n)), \quad b_n = q_0 - \hat{q}_{cn} \quad (4.16)$$

Below we need the identities and the asymptotics as $n \rightarrow \infty$ from [KK]:

$$(-1)^{n+1} i \sin k(z) = \sinh v(z) = \pm |\Delta^2(z) - 1|^{\frac{1}{2}} > 0 \quad \text{all } z \in g_n^\pm, \quad (4.17)$$

$$v(z) = \pm |(z - e_n^-)(e_n^+ - z)|^{\frac{1}{2}} (1 + O(n^{-2})), \quad \sinh v(z) = v(z)(1 + O(|g_n|^2)), \quad z \in \bar{g}_n^\pm. \quad (4.18)$$

We rewrite the equation $\Psi_0^+ = 0$ in the form $\varphi_1 Y_1 = -i \sin k \tilde{\varphi}(z_n)$. Then we obtain

$$\begin{aligned} 2\delta \varepsilon_n (1 + O(\varepsilon_n)) &= v(z) 2\varepsilon_n^2 (b_n + O(\varepsilon_n)) = \sqrt{\delta(|g_n| - \delta)} 2\varepsilon_n^2 (b_n + O(\varepsilon_n)), \\ \sqrt{\delta} &= \varepsilon_n \sqrt{|g_n| - \delta} (b_n + O(\varepsilon_n)), \quad \delta = z_n - \mu_n, \end{aligned}$$

where $\sqrt{\delta} > 0$ if $b_n > 0$ and $\sqrt{\delta} < 0$ if $b_n < 0$. Then last asymptotics imply $\delta = \varepsilon_n^2 |g_n| (b_n + O(\varepsilon_n))^2$, where $b_n = q_0 - \widehat{q}_{cn}$, $\varepsilon_n = \frac{1}{2\pi n}$, which yields (1.16).

ii) Let $q \in \mathcal{Q}_t$, $q_0 = 0$ and let each $|\widehat{q}_{cn}| > n^{-\alpha}$ for some $\alpha \in (0, 1)$ and for n large enough. The proof of other cases is similar. Then using the inverse spectral theory from [K5], see page 3, for any sequence $\varkappa = (\varkappa_n)_1^\infty \in \ell^2$, $\varkappa_n \geq 0$ there exists a potential $p \in L^2(0, 1)$ such that each gap length $|\gamma_n| = \varkappa_n$, $n \geq 1$ for n large enough. Moreover, for n large enough we can the gap in the form $\gamma_n = (E_n^-, E_n^+)$, where $\mu_n^2 = E_n^-$ or $\mu_n^2 = E_n^+$. In order to choose $\mu_n^2 = E_n^-$ or $\mu_n^2 = E_n^+$ we do the following. For any sequence $\sigma = (\sigma_n)_1^\infty$, where $\sigma_n \in \{0, 1\}$, using Theorem 1.3 (i) we obtain:

If $\sigma_n = 1$ and $\widehat{q}_{cn} < -n^{-\alpha}$ (or $\widehat{q}_{cn} > n^{-\alpha}$), then taking $\mu_n^2 = E_n^-$ (or $\mu_n^2 = E_n^+$) we deduce that λ_n is an eigenvalue for n large enough.

If $\sigma_n = 0$ and $\widehat{q}_{cn} > n^{-\alpha}$ (or $\widehat{q}_{cn} < -n^{-\alpha}$), then taking $\mu_n^2 = E_n^-$ (or $\mu_n^2 = E_n^+$) we deduce that λ_n is an antibound state for n large enough.

iii) Let $p \in L^2(0, 1)$ such that $\gamma_n = (E_n^-, E_n^+)$, where $\mu_n^2 = E_n^-$ or $\mu_n^2 = E_n^+$ for $n \in \mathbb{N}_0$ large enough. Let $\sigma = (\sigma_n)_1^\infty$ be any sequence, where $\sigma_n \in \{0, 1\}$. We take $|\widehat{q}_{cn}| > n^{-\alpha}$ for $n \in \mathbb{N}_0$ large enough. We need to choose the sign of q_{cn} . Using Theorem 1.3 (i) we take the sign of q_{cn} by

If $\sigma_n = 0$ and $\lambda_n^0 = E_n^-$ (or $\lambda_n^0 = E_n^+$), then taking $\widehat{q}_{cn} > n^{-\alpha}$ (or $\widehat{q}_{cn} < -n^{-\alpha}$) we deduce that λ_n is an antibound state.

If $\sigma_n = 1$ and $\lambda_n^0 = E_n^-$ (or $\lambda_n^0 = E_n^+$), then taking $\widehat{q}_{cn} < -n^{-\alpha}$ (or $\widehat{q}_{cn} > n^{-\alpha}$) we deduce that λ_n is an eigenvalue. ■

Proof of Corollary 1.5. Let $\#_{bs}(H, \Omega)$ (or $\#_{abs}(H, \Omega)$) be the total number of bound states (or anti bound states) of H on the segment $\Omega \subset \gamma_n^{(1)}$ (or $\Omega \subset \gamma_n^{(2)}$) for some $n \geq 0$. Here each state being counted according to its multiplicity.

Recall that $H_\tau = H_0 + q_\tau$, where $q_\tau = q(\frac{x}{\tau})$ and $\tau \rightarrow \infty$. Let $\Omega = [E_1, E_2] \subset \overline{\gamma_n^{(1)}}$ for some $n \geq 0$. Then using the result of Sobolev [So] with a modification of Schmidt [Sc] we obtain

$$\#_{bs}(H_\tau, \Omega) = \tau \int_0^\infty \left(\rho(E_2 - q(x)) - \rho(E_1 - q(x)) \right) dx + o(\tau) \quad \text{as } \tau \rightarrow \infty. \quad (4.19)$$

Theorem 1.2 (iii) implies $\#_{abs}(H_\tau, \Omega^{(2)}) \geq 1 + \#_{bs}(H_\tau, \Omega)$, which together with (4.19) yield (1.18). ■

Acknowledgments. The research was partially supported by EPSRC grant EP/D054621. The various parts of this paper were written at ESI, Vienna, Université de Genève, Section de Mathématiques and Mathematical Institute of the Tsukuba Univ., Japan. The author is grateful to the Institutes for the hospitality. The author would like also to thank A. Sobolev (London) and K. Schmidt (Cardiff) for useful discussions about the asymptotics associated with Corollary 1.5.

REFERENCES

- [BKW] Brown, B.; Knowles, I.; Weikard, R. On the inverse resonance problem, J. London Math. Soc. (2) 68 (2003), no. 2, 383–401.
- [BW] Brown, B.; Weikard, R. The inverse resonance problem for perturbations of algebro-geometric potentials. Inverse Problems 20 (2004), no. 2, 481–494.
- [Ch] Christiansen, T. Resonances for steplike potentials: forward and inverse results. Trans. Amer. Math. Soc. 358 (2006), no. 5, 2071–2089.
- [CL] Coddington, E.; Levinson, N. Theory of ordinary differential equations. New York, Toronto, London: McGraw-Hill 1955.

- [E] Eastham M. The spectral theory of periodic differential equations. Scottish Academic Press, Edinburg, 1973.
- [F1] Firsova, N. Resonances of the perturbed Hill operator with exponentially decreasing extrinsic potential. *Mat. Zametki* 36 (1984), 711–724.
- [F2] Firsova, N. The Levinson formula for a perturbed Hill operator. (Russian) *Teoret. Mat. Fiz.* 62 (1985), no. 2, 196–209.
- [F3] Firsova, N. A direct and inverse scattering problem for a one-dimensional perturbed Hill operator. *Mat. Sb.* 130(172) (1986), no. 3, 349–385.
- [F4] Firsova, N. A trace formula for a perturbed one-dimensional Schrödinger operator with a periodic potential. I. (Russian) *Problems in mathematical physics*, No. 7 (Russian), pp. 162–177, Izdat. Leningrad. Univ., Leningrad, 1974
- [Fr] Froese, R. Asymptotic distribution of resonances in one dimension, *J. Diff. Eq.*, 137(1997), 251–272.
- [GT] Garnett, J.; Trubowitz, E. Gaps and bands of one dimensional periodic Schrödinger operators II. *Comment. Math. Helv.* 62(1987), 18–37.
- [GS] Gesztesy, F.; Simon, B. A short proof of Zheludev’s theorem. *Trans. Amer. Math. Soc.* 335 (1993), no. 1, 329–340.
- [HKS] Hinton, D.; Klaus, M.; Shaw, J. On the Titchmarsh-Weyl function for the half-line perturbed periodic Hills equation. *Quart. J. Math. Oxford Ser. (2)* 41 (1990), no. 162, 189–224.
- [H] Hitrik, M. Bounds on scattering poles in one dimension. *Comm. Math. Phys.* 208 (1999), no. 2, 381–411.
- [IM] Its, A.; Matveev, V. Schrödinger operators with the finite-gap spectrum and the N-soliton solutions of the Korteweg de Fries equation. *Teoret. Math. Phys.* 23(1975), 51–68.
- [J] Javrlan, V. Some perturbations of self-adjoint operators. (Russian) *Akad. Nauk Armjan. SSR Dokl.* 38 (1964), 3–7.
- [KK] Kargaev, P.; Korotyaev, E. Effective masses and conformal mappings. *Comm. Math. Phys.* 169 (1995), no. 3, 597–625.
- [KK1] Kargaev, P.; Korotyaev, E. Inverse Problem for the Hill Operator, the Direct Approach. *Invent. Math.*, 129(1997), no. 3, 567–593.
- [KM] Klopp, F.; Marx, M. The width of resonances for slowly varying perturbations of one-dimensional periodic Schrödingers operators, *Seminaire: Equations aux Derivées Partielles*. 2005–2006, Exp. No. IV, 18 pp., *Semin. Equ. Deriv. Partielles*, Ecole Polytech., Palaiseau.
- [K1] Korotyaev, E. Inverse resonance scattering on the half line, *Asymptotic Anal.* 37(2004), No 3/4, 215–226.
- [K2] Korotyaev, E. Inverse resonance scattering on the real line. *Inverse Problems* 21 (2005), no. 1, 325–341.
- [K3] Korotyaev, E. Stability for inverse resonance problem. *Int. Math. Res. Not.* 2004, no. 73, 3927–3936.
- [K4] Korotyaev, E. 1D Schrödinger operator with periodic plus compactly supported potentials, preprint 2009, arXiv:0904.2871.
- [K5] Korotyaev, E. Inverse problem and the trace formula for the Hill operator. II *Math. Z.* 231 (1999), no. 2, 345–368.
- [K6] Korotyaev, E. Characterization of the spectrum of Schrödinger operators with periodic distributions. *Int. Math. Res. Not.* 2003, no. 37, 2019–2031.
- [K7] Korotyaev, E. Estimates of periodic potentials in terms of gap lengths. *Comm. Math. Phys.* 197 (1998), no. 3, 521–526.
- [Koo] Koosis, P. The logarithmic integral I, Cambridge Univ. Press, Cambridge, London, New York 1988.
- [M] Marchenko, V. Sturm-Liouville operator and applications. Basel: Birkhäuser 1986.
- [MO] Marchenko, V.; Ostrovski I. A characterization of the spectrum of the Hill operator. *Math. USSR Sbornik* 26(1975), 493–554.
- [N-Z] Novikov, S.; Manakov, S. V.; Pitaevskii, L. P.; Zakharov, V. E. Theory of solitons. The inverse scattering method. Translated from the Russian. *Contemporary Soviet Mathematics*. Consultants Bureau [Plenum], New York, 1984.
- [PT] Pöschel, P.; Trubowitz, E. Inverse Spectral Theory. Boston: Academic Press, 1987.

- [Rb] Rofe-Beketov, F. A finiteness test for the number of discrete levels which can be introduced into the gaps of the continuous spectrum by perturbations of a periodic potential. Dokl. Akad. Nauk SSSR 156 (1964), 515–518.
- [Sc] Schmidt, K. M. Eigenvalue asymptotics of perturbed periodic Dirac systems in the slow-decay limit. Proc. Amer. Math. Soc. 131 (2003) 1205–1214.
- [S] Simon, B. Resonances in one dimension and Fredholm determinants. J. Funct. Anal. 178(2000), no. 2, 396–420.
- [So] Sobolev, A.V. Weyl asymptotics for the discrete spectrum of the perturbed Hill operator. Estimates and asymptotics for discrete spectra of integral and differential equations (Leningrad, 1989–90), 159–178, Adv. Soviet Math., 7, Amer. Math. Soc., Providence, RI, 1991.
- [T] Titchmarsh, E. Eigenfunction expansions associated with second-order differential equations 2, Clarendon Press, Oxford, 1958.
- [Tr] Trubowitz, E. The inverse problem for periodic potentials. Commun. Pure Appl. Math. 30(1977), 321–337.
- [Ja] Javrjan, V. A. Some perturbations of self-adjoint operators. (Russian) Akad. Nauk Armjan. SSR Dokl. 38 1964 3–7.
- [Z] Zworski, M. Distribution of poles for scattering on the real line, J. Funct. Anal. 73(1987), 277–296.
- [Z1] Zworski, M. SIAM, J. Math. Analysis, "A remark on isopolar potentials" 82(6), 2002, 1823–1826.
- [Z2] Zworski, M. Counting scattering poles. In: Spectral and scattering theory (Sanda, 1992), 301–331, Lecture Notes in Pure and Appl. Math., 161, Dekker, New York, 1994.
- [Zh1] Zheludev, V. A. The eigenvalues of a perturbed Schrödinger operator with periodic potential. (Russian) 1967 Problems of Mathematical Physics, No. 2, Spectral Theory, Diffraction Problems pp. 108–123.
- [Zh2] Zheludev, V. A. The perturbation of the spectrum of the Schrödinger operator with a complex-valued periodic potential. (Russian) Problems of mathematical physics, Spectral theory 3(1968), 31–48.
- [Zh3] Zheludev, V. The spectrum of Schrödinger's operator, with a periodic potential, defined on the half-axis. Works of Dep. of Math. Analysis of Kaliningrad State University (1969) (Russian), pp. 18–37.

SCHOOL OF MATH., CARDIFF UNIVERSITY. SENGHENNYDD ROAD, CF24 4AG CARDIFF, WALES, UK.
 EMAIL: KOROTYAEV@CF.AC.UK